

# Gamma Tilting Calculus for GGC and Dirichlet means with applications to Linnik processes and Occupation Time Laws for Randomly Skewed Bessel Processes and Bridges.

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This paper explores various interfaces between the class of generalized gamma convolution (GGC) random variables, Dirichlet process mean functionals and phenomena connected to the local time and occupation time of  $p$ -skew Bessel processes and bridges discussed in Barlow, Pitman and Yor (1989), Pitman and Yor (1992, 1997). First, some general calculus for GGC and Dirichlet process means functionals is developed. It then proceeds, via an investigation of positive Linnik random variables, and more generally random variables derived from compositions of a stable subordinator with GGC subordinators, to establish various distributional equivalences between these models and phenomena connected to local times and occupation times of what are defined as randomly skewed Bessel processes and bridges. This yields a host of interesting identities and explicit density formula for these models. Randomly skewed Bessel processes and bridges may be seen as a randomization of their  $p$ -skewed counterparts, and are shown to naturally arise via exponential tilting. As a special result it is shown that the occupation time of a  $p$ -skewed random Bessel process or (generalized) bridge is equivalent in distribution to the occupation time of a non-trivial randomly skewed process.

## 1 Introduction

This paper explores various interfaces between the class of generalized gamma convolution (GGC) random variables, Dirichlet process mean functionals and phenomena connected to the local time and occupation time of skew Bessel processes and bridges discussed in Barlow, Pitman and Yor (1989), Pitman and Yor (1992, 1997b). The latter concepts are also related to the now ubiquitous two-parameter Poisson-Dirichlet models, see for instance Pitman and Yor (1997a) and Pitman (2006). Specifically, we take the viewpoint as in James (2006), that GGC random variables can be represented as linear functionals of Gamma processes and hence there is a strong link to the distributional theory of Dirichlet mean functionals developed in Cifarelli and Regazzini (1990). Here, we first develop two types of distributional results, one is a simple but quite useful result involving scaling general Dirichlet mean functionals by beta random variables, and the other is a form of gamma tilting calculus. We then investigate the class of positive Linnik random variables (see Devroye (1996, 1990)), establishing new distributional identities, and a link to local and occupation times of skew Bessel processes and bridges. As a natural extension, we investigate compositions of a stable subordinator and a subordinator which has a GGC law. We use this to develop additionally a scaling calculus for (generalized) Dirichlet mean functionals. These points then lead to the development of a series of interesting results, including explicit density formula, for what we call randomly skewed Bessel processes and bridges. As a special case, we establish equivalences in law between occupation times of  $p$ -skewed Bessel processes and bridges and non-trivial randomly skewed processes. Some recent

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work related to our line of investigation include, James (2006), James, Lijoi and Prünster (2006), Bertoin, Fujita, Roynette and Yor (2006) and Fujita and Yor (2006). With some further reference to these latter two works, we note that a key result in this manuscript is to show that the random variable  $X_\alpha = S_\alpha/S'_\alpha$ , where  $S_\alpha$  and  $S'_\alpha$  are iid  $\alpha$ -stable random variables is indeed a Dirichlet mean functional. The explicit distribution of  $X_\alpha$  was obtained in Lamperti (1958) and, as shall be shown, this fact combined with the correspondence to Dirichlet mean functionals allows one to easily obtain explicit densities for a host of models connected to Bessel phenomena. For some other references relevant to skew Bessel processes see Bertoin and Yor (1996), Kasahara and Watanabe (2005) and Watanabe (1995).

## 2 The relationship between GGC, Gamma processes and Dirichlet mean functionals

A positive infinitely divisible random variable  $Z_\theta$  is said to be a generalized Gamma convolution (GGC) if for positive  $\lambda$  one can write its Laplace transform in the form

$$\mathbb{E}[e^{-\lambda Z_\theta}] = e^{-\theta \vartheta(\lambda)}$$

where for a sigma-finite measure  $\nu$  on  $(0, \infty)$ ,

$$\vartheta(\lambda) = \int_0^\infty \log(1 + \lambda x) \nu(dx) = \int_0^\infty \int_0^\infty (1 - e^{-s\lambda}) s^{-1} e^{-s/x} \nu(dx).$$

$\nu$  is chosen so that  $\vartheta(\lambda) < \infty$ , and is often referred to as a Thorin measure. For precise conditions on  $\nu$  see Bondesson (1992). The corresponding Lévy measure of  $Z_\theta$  is,

$$\theta s^{-1} \int_0^\infty e^{-s/x} \nu(dx).$$

We will say  $Z_\theta$  is  $\text{GGC}(\theta, \nu)$ . Naturally every GGC can be used to generate a subordinator which we denote as  $Z_\theta(t)$  for each  $t \geq 0$ . We will also be interested in the special case where  $\nu = H$  is a probability measure, in that case we write  $\text{GGC}(\theta, H)$  and will sometimes refer to  $Z_\theta$  as an FGGC or finite GGC, a term we used in James (2006). As in that work, we will exploit a close connection between GGC, and Gamma processes, and in the case of FGGC, its connection with the Dirichlet process. [See Ferguson (1973) for the Dirichlet process and its use in Bayesian nonparametric statistics]. Now let  $\Gamma_{\theta\nu}$  denote a gamma process on  $(0, \infty)$  with shape parameter  $(\theta\nu)$ , this is a completely random measure whose law is characterized by its Laplace functional for some positive function  $g$  as

$$\mathbb{E}[e^{-\lambda \Gamma_{\theta\nu}(g)}] = e^{-\theta \int_0^\infty \log(1 + \lambda g(x)) \nu(dx)},$$

where  $\Gamma_{\theta\nu}(g) := \int_0^\infty g(x) \Gamma_{\theta\nu}(dx)$ . Hence setting  $g(x) = x$ , we see that every  $\text{GGC}(\theta, \nu)$  random variable can be written as a mean (linear) functional of  $\Gamma_{\theta\nu}$ . Let  $(J_{k,\theta})$  denote the ranked jump sizes of a subordinator based on a  $\text{GAMMA}(\theta)$  distribution, such that  $\sum_{k=1}^\infty J_{k,\theta} = G_\theta$ . Now recall that for each  $\theta > 0$ ,  $G_\theta$ , which is a  $\text{GAMMA}(\theta)$  random variable, is independent of the sequence of ranked probabilities  $(\tilde{P}_{k,\theta} = J_{k,\theta}/G_\theta)$ . The law of the sequence  $(\tilde{P}_{k,\theta})$  is sometimes referred to as the Poisson-Dirichlet distribution and, using notation from Pitman and Yor (1997a), is denoted  $\text{PD}(0, \theta)$ . These points imply that

$$\int_0^\infty x \Gamma_{\theta\nu}(dx) \stackrel{d}{=} G_\theta M_\theta(\nu)$$

where  $M_\theta(\nu) \stackrel{d}{=} \sum_{k=1}^\infty \tilde{P}_{k,\theta} W_k$ , where the  $(W_k)$  are independent of  $(J_{k,\theta})$  and are the points of a Poisson random measure with mean intensity  $\nu$ . Furthermore, the distribution of  $M_\theta(\nu)$  is characterized by its Cauchy-Stieltjes transform of order  $\theta$ , which is

$$\mathbb{E}[(1 + \lambda M_\theta(\nu))^{-\theta}] = \mathbb{E}[e^{-\lambda G_\theta M_\theta(\nu)}] = e^{-\theta \vartheta(\lambda)}.$$

Hence, importantly, it follows that  $Z_\theta \stackrel{d}{=} G_\theta M_\theta(\nu)$ . Now when  $\nu = H$  we may define a Dirichlet process with shape parameter  $\theta H$  as  $P_{0,\theta}(\cdot) = \sum_{k=1}^{\infty} \tilde{P}_{k,\theta} \delta_{Z_k}(\cdot)$ , where the  $(Z_k)$  are iid with common distribution  $H$ . Then a Dirichlet mean functional is defined as

$$M_\theta(H) \stackrel{d}{=} \int_0^\infty x P_{0,\theta}(dx).$$

Hence, if  $Z_\theta$  is a GGC( $\theta, H$ ) then  $Z_\theta \stackrel{d}{=} G_\theta M_\theta(H)$ .

REMARK 1. Note of course that when  $\nu$  has infinite mass the use of  $\theta$  is somewhat redundant. However, it will play an important role as the decomposition  $G_\theta M_\theta(\nu)$  always makes sense.

REMARK 2. In Bondesson (1992) the Lévy exponent of a GGC is represented in a slightly different, but certainly equivalent, way as

$$\vartheta(\lambda) = \int_0^\infty \log(1 + \lambda/y) U(dy).$$

where  $U$  is some sigma-finite measure. In this work, we are using a representation which more closely matches that used in the literature on Dirichlet process mean functionals.

REMARK 3. See Vershik, Yor and Tsilevich (2001) and James (2005) for some uses of the decomposition  $G_\theta M_\theta(H)$

REMARK 4. Throughout, for  $0 < \alpha < 1$ , we will define  $S_\alpha$  to be a unilateral  $(\alpha)$ -stable random variable having the Laplace transform

$$\mathbb{E}[e^{-\lambda S_\alpha}] = e^{-\lambda^\alpha}.$$

This random variable is well known to be a GGC with an infinite Thorin measure. See example 3.2.1 of Bondesson (1992).

### 3 Some calculus for GGC/Dirichlet mean functionals

Suppose that  $X$  has distribution  $H$ , and define the function

$$\Phi(t) = \int_0^\infty \log(|t - x|) I(t \neq x) H(dx) = \mathbb{E}[\log(|t - X|) I(t \neq X)]$$

furthermore, letting  $H(t) = \int_0^t H(dx)$ , define

$$\Delta_\theta(t|H) = \frac{1}{\pi} \sin(\pi \theta H(t)) e^{-\theta \Phi(t)}.$$

Cifarelli and Regazzini (1990, 1994), see also Cifarelli and Mellili (2000), apply inversion formula to obtain the distributional formula for  $M_\theta(H)$  as follows. For all  $\theta > 0$ , the cdf can be expressed as

$$(1) \quad \int_0^x (x - t)^{\theta-1} \Delta_\theta(t|H) dt$$

provided that  $\theta H$  possesses no jumps of size greater than or equal to one. If we let  $f_{M_\theta}(\cdot|H)$  denote the density of  $M_\theta(H)$ , it takes its simplest form for  $\theta = 1$ , which is

$$(2) \quad f_{M_1}(x|H) = \Delta_1(x|H) = \frac{1}{\pi} \sin(\pi H(x)) e^{-\Phi(x)}.$$

Formula for  $\theta > 1$  are also available from Cifarelli and Regazzini (1990,1994) and Cifarelli and Mellili (2000). However, we shall not explicitly use this and instead give an expression for the density which holds for all  $\theta > 0$ , recently obtained by James, Lijoi and Prünster (2006), as follows,

$$(3) \quad f_{M_\theta}(x|H) = \int_0^x (x-t)^{\theta-1} d_\theta(t|H) dt$$

where

$$d_\theta(t|H) = \frac{d}{dt} \sin(\pi\theta H(t)) e^{-\theta\Phi(t)}.$$

We do point out, that except for some of the very recent work we mention here, there are very few examples of mean functionals where the explicit density has been calculated. This is due in part to the fact that it is not necessarily obvious how to write  $\Delta_\theta(t|H)$  in a nice form, or otherwise how to calculate  $\Phi(x)$ . Our results in this section provide some additional tools to obtain more explicit expressions and also point to apparently unknown inter-relationships between different mean functionals which occurs through a tilting operation.

The first result, in Theorem 3.1 below, shows that multiplying a mean functional by a certain beta random variable may lead to more simplified expressions. Additionally, we point out that this result was constructed in an effort to better understand the result in Proposition 15 of Pitman and Yor (1997b) and, as we show, paves the way for other interesting results related to occupation times and local times of Bessel processes.

**REMARK 5.** First, we introduce a bit more notation. The notation  $G_a$ , for  $a > 0$ , will denote a Gamma random variable with shape  $a$  and scale 1, with law denoted as  $\text{GAMMA}(a)$ . Throughout we will use the notation  $B_{a,b}$  to denote a Beta random variable with parameters  $(a,b)$ , and write  $\text{BETA}(a,b)$  to denote this law. Additionally, unless otherwise specified, if for random variables  $X$  and  $Y$  we write the product  $XY$ , it will be assumed that  $X$  and  $Y$  are independent. Additionally the notation  $X'$  means that  $X'$  is an independent random variable with the same distribution as  $X$ .

**Theorem 3.1** *Let  $X$  denote a random variable with distribution  $H$ . For  $0 < p \leq 1$ , let  $Y_p$  denote a  $\text{BERNOULLI}(p)$  random variable independent of  $X$ . Then the distribution of  $XY_p$  is given by  $H^{(p)}(dv) := pH(dv) + (1-p)\delta_0(dv)$ . Now define  $\Phi(t) := \mathbb{E}[\log(|t-X|)\mathbb{I}(X \neq t)]$ . Hence  $\Phi^{(p)}(t) := \mathbb{E}[\log(|t-XY_p|)\mathbb{I}(XY_p \neq t)] = p\Phi(t) + (1-p)\log(t)\mathbb{I}(t \neq 0)$ . Then for  $\theta > 0$ ,*

$$(4) \quad M_\theta(H^{(p)}) \stackrel{d}{=} M_{p\theta}(H) B_{\theta p, \theta(1-p)}$$

and  $G_\theta M_\theta(H^{(p)}) \stackrel{d}{=} G_{\theta p} M_{\theta p}(H)$ . Hence  $\text{GGC}(\theta, H^{(p)}) = \text{GGC}(\theta p, H)$  for all  $\theta > 0$  and  $0 < p < 1$ . As special cases,

$$(i) \quad M_1(H^{(p)}) = M_p(H) B_{p, 1-p}, \text{ with density, for } 0 < p < 1,$$

$$(5) \quad \frac{1}{\pi} x^{-(1-p)} \sin(\pi p[1-H(x)]) e^{-p\Phi(x)}$$

$$(ii) \quad M_{\frac{1}{p}}(H^{(p)}) = M_1(H) B_{1, \frac{(1-p)}{p}}.$$

**PROOF.** First note the distributional identity,  $G_{\theta p} \stackrel{d}{=} G_\theta B_{\theta p, \theta(1-p)}$ , for all  $\theta > 0$ . Then for (4) it suffices to check the equivalence  $G_\theta M_\theta(H^{(p)}) \stackrel{d}{=} G_{\theta p} M_{\theta p}(H)$ . But this is obvious upon taking Laplace transforms. The density in (5) follows as a special case of (2). Note additionally the identity  $\sin(\pi[pH(x) + 1 - p]) = \sin(\pi[p(1-H(x))])$ , for  $0 < p < 1$ .  $\square$

### 3.0.1 Remark about obtaining the distribution of a FGGC subordinator for all time points

It is well known that if  $\{Z(t), t \geq 0\}$  is some subordinator, even if we have an explicit expression for its density or distribution at time say 1, i.e. the random variable  $Z(1)$ , it is not obvious how to find explicit densities or distribution functions for general  $Z(t)$ , at fixed time points  $t$ . Of course, by the convolution properties of infinitely divisible random variables one can describe the law of a random variable for  $Z(t)$  for  $t > 1$ , in terms of a linear combination of independent random variables based on  $\{Z(s) : s \leq 1\}$ . Hence, from this point of view, one may simply concentrate on obtaining explicit expressions for the density of random variables for  $Z(t)$ , where  $t$  is some fixed value in  $(0, 1]$ . In the case where  $\{Z_\theta(t); t \geq 0\}$  is a FGGC subordinator, one may obtain an explicit description of its density for each fixed  $0 < t \leq 1$ , provided that  $\Phi$  has a tractable form, using Theorem 3.1. Formally if  $0 < \theta t < 1$ , it follows that

$$Z_\theta(t) \stackrel{d}{=} G_{\theta t} M_{\theta t}(H) \stackrel{d}{=} G_1 B_{\theta t, 1-\theta t} M_{\theta t}(H) \stackrel{d}{=} G_1 M_1(H^{(\theta t)})$$

where  $M_1(H^{(\theta t)})$  has a density (of non-integral form),

$$\frac{1}{\pi} x^{\theta t - 1} \sin(\pi \theta t [1 - H(x)]) e^{-\theta t \Phi(x)}.$$

The point here is that density of  $M_1(H^{(\theta t)})$  generally has a simpler description than that of  $M_{\theta t}(H)$ . Naturally, these models agree when  $\theta t = 1$ .

REMARK 6. These points and Theorem 3.1 are connected to an expression for a GGC( $\beta, H$ ) density for  $0 < \beta < 1$  given in Bondesson (1992, p. 37-38). Relationships to beta random variables are not noted there.

## 3.1 Some basic exponential tilting calculus for Gamma mixtures

Here we develop distributional results for exponential tilting of scale mixtures of Gamma random variables. We then specify this to the case of exponential tilting of GGC models which always can be written in this form. An important point is that we are able to establish a clear distributional link between a  $M_\theta(\nu)$  and a corresponding random variable obtained from tilting  $G_\theta M_\theta(\nu)$ , which, as a by-product, has new implications for the study of Dirichlet mean functionals. These results will play a fundamental role throughout the text.

**Theorem 3.2** *Suppose that  $W = G_\theta M$  where  $M$  and  $G_\theta$  are independent positive random variables and  $G_\theta$  is GAMMA( $\theta$ ). Then for  $b$  and  $c$  positive numbers, the random variable,  $\tilde{W}_\theta$  with density  $e^{-c/bw} b^{-1} f_W(w/b) / E[e^{-cG_\theta M}]$  satisfies,*

$$\tilde{W}_\theta \stackrel{d}{=} (b/c) G_\theta Y_{\theta, c}$$

where,

(i)  $Y_{\theta, c}$  is a random variable with density

$$\frac{f_Y(y) [(1-y)]^\theta}{\mathbb{E}[(1+cM)^{-\theta}]},$$

where  $Y \stackrel{d}{=} cM/(cM+1)$ , and  $\mathbb{E}[(1+cM)^{-\theta}] = \mathbb{E}[e^{-cG_\theta M}]$ .

(ii) *Equivalently the distribution of  $Y_{\theta, c}$  is the distribution of  $cM/(cM+1)$  taken with respect to the density  $(cm+1)^{-\theta} f_M(m) / \mathbb{E}[e^{-cG_\theta M}]$ .*

(iii) Conversely  $f_M(x) = (1+x)^{\theta-2} f_{Y_{\theta,1}}(\frac{x}{1+x}) \mathbb{E}[(1+M)^{-\theta}]$

PROOF. Let us proceed by checking Laplace and Cauchy transforms. By direct argument, working with the density, the Laplace transform of  $\tilde{W}_\theta$  evaluated at  $\lambda$  is,

$$\frac{\mathbb{E}[(1+(c+\lambda b)M)^{-\theta}]}{\mathbb{E}[(1+cM)^{-\theta}]}$$

which, for  $c \neq 0$ , may be re-written as

$$\frac{\mathbb{E}[(1+c(1+\lambda(b/c))M)^{-\theta}]}{\mathbb{E}[(1+cM)^{-\theta}]}$$

Now for simplicity we set  $b/c = 1$  and note that the Laplace transform of  $G_\theta Y_{\theta,c}$  is, by definition of  $Y_{\theta,c}$ ,

$$\mathbb{E}[(1+\lambda Y_{\theta,c})^{-\theta}] = \int_0^\infty \left(1 + \lambda \frac{cx}{cx+1}\right)^{-\theta} \frac{(1+cx)^{-\theta} f_M(x)}{\mathbb{E}[(1+cM)^{-\theta}]} dx$$

Simple algebra completes the result.  $\square$

The next two results are specialized to GGC.

**Proposition 3.1** *Let  $L$  be a  $GGC(\theta, \nu)$  random variable with density denoted  $f_L$ . Hence  $L = G_\theta M_\theta(\nu)$ . Now define a tilted random variable  $\tilde{L}$  having density proportional to  $e^{-c/bw} b^{-1} f_W(w/b)$  for  $c \geq 0$  and  $b > 0$ . Denote this law of  $\tilde{L}$  as  $GGC^{(b,c)}(\theta, \nu)$ . It then follows that for  $c > 0$   $\tilde{L} \stackrel{d}{=} (b/c)L_c$  where  $L_c$  is  $GGC^{(c,c)}(\theta, \nu)$ .*

(i) The Lévy measure of  $\tilde{L}$  is

$$s^{-1} e^{-(c/b)s} \int_0^\infty e^{-s/bw} \nu(dw)$$

(ii) When  $c = 0$ ,  $\tilde{L} = bL$ . Otherwise for  $c > 0$ ,  $\tilde{L} \stackrel{d}{=} (b/c)G_\theta M_\theta(\nu^{(c,c)})$ . The Thorin measure  $\nu^{(c,c)}$  is determined by  $\nu$  as indicated by the following expression for the Lévy density of  $G_\theta M_\theta(\nu^{(c,c)})$ ,

$$s^{-1} \int_0^\infty e^{-s \frac{cr+1}{cr}} \nu(dr) = s^{-1} \int_0^\infty e^{-s/y} \nu^{(c,c)}(dy)$$

(iii) Referring to Theorem 3.2,  $M_\theta(\nu^{(c,c)}) \stackrel{d}{=} Y_{\theta,c}$ , for  $M \stackrel{d}{=} M_\theta(\nu)$ .

PROOF. The first statement is just a standard result for exponentially tilting infinitely divisible random variables coupled with the specific form of a GGC Lévy measure. Statement [(ii)] is just algebra, which yields importantly the representation  $G_\theta M_\theta(\nu^{(c,c)})$ . Statement [(iii)] then follows from Theorem 3.2  $\square$

We close this section with an important variation in the case where  $\nu$  is a probability measure.

**Proposition 3.2** *Let  $L$  be a  $GGC(\theta, H)$ , then there exists a random variable  $X$  with distribution  $H$  and a random variable  $A_c^* = cX/cX + 1$  on  $[0, 1]$  such that if  $L_c$  is now  $GGC^{(c,c)}(\theta, H)$  it is equivalently  $GGC(\theta, Q_c)$ , where  $Q_c$  is the distribution of  $A_c^*$  derived from  $H$  and is a special case of  $\nu^{(c,c)}$ . In other words the Lévy measure of  $L_c$  may be written as*

$$\theta s^{-1} \mathbb{E}[e^{-s/A_c^*}]$$

Additionally the following distributional relationships hold.

- (i) Suppose that the density of  $M_\theta(H)$ , say  $f_{M_\theta}(\cdot|H)$  is known. Then the density of  $M_\theta(Q_c)$  is expressible as

$$f_{M_\theta}(y|Q_c) = \frac{(1-y)^{\theta-2}}{c\mathbb{E}[(1+cM_\theta(H))^{-\theta}]} f_{M_\theta}\left(\frac{y}{c(1-y)}|H\right)$$

- (ii) Conversely, if the density of  $M_\theta(Q_c)$ ,  $f_{M_\theta}(\cdot|Q_c)$ , is known then the density of  $M_\theta(H)$  is given by

$$f_{M_\theta}(x|H) = (1+x)^{\theta-2} f_{M_\theta}\left(\frac{x}{1+x}|Q_1\right) \mathbb{E}[(1+M_\theta(H))^{-\theta}]$$

REMARK 7. The gamma exponential tilting operation is quite special and we point out that for a fixed value of  $\theta$ ,  $Y_{\theta,c}$  cannot achieve all possible distributions on  $[0, 1]$ . So for example when  $\theta = 1$ ,  $Y_{1,1}$  cannot be UNIFORM $[0, 1]$ . This is equivalent to noting that a density on  $[0, 1]$  of the form

$$f_Y(y) \propto \frac{1}{(1-y)}$$

does not exist. In general, the UNIFORM $[0, 1]$  is possible for  $Y_{\theta,c}$ , only when  $0 < \theta < 1$ .

### 3.2 An example with some connections to the occupation time of skew Brownian bridge

Set  $H = U[0, 1]$  to denote that the corresponding random variable  $X$  is UNIFORM $[0, 1]$ . It is known from Diaconis and Kemperman (1994) that the density of  $M_1(U[0, 1])$  is

$$(6) \quad \frac{e}{\pi} \sin(\pi y) y^{-y} (1-y)^{-(1-y)} \text{ for } 0 < y < 1.$$

Note furthermore that  $W = G_1 M_1(U[0, 1])$  is GGC $(1, U[0, 1])$  and has a rather strange Laplace transform,

$$\mathbb{E}[e^{-\lambda G_1 M_1(U[0, 1])}] = e(1+\lambda)^{-\left(\frac{\lambda+1}{\lambda}\right)}.$$

We can use this fact combined with the previous results to obtain a new explicit expression for the density of what we believe should be an important mean functional and corresponding infinitely divisible random variable.

**Proposition 3.3** *Let  $G_1/E$  be the ratio of two independent exponential (1) random variables having density  $\zeta(dx)/dx = (1+x)^{-2}$  for  $x > 0$ . Now let  $L$  denote a GGC $(1, \zeta)$  random variable, with log Laplace transform  $\log \mathbb{E}[e^{-\lambda G_1 M_1(\zeta)}] = -\frac{\lambda}{\lambda-1} \log(\lambda)$ . Then equivalently the Lévy measure of  $L$  is given by*

$$s^{-1} \mathbb{E}[e^{-s \frac{E}{G_1}}] = s^{-1} \mathbb{E}[e^{-s \frac{G_1}{E}}]$$

Furthermore  $L \stackrel{d}{=} G_1 M_1(\zeta)$ , where  $M_1(\zeta)$  has density,

$$f_{M_1}(x|\zeta) = \frac{1}{\pi} \sin(\pi \frac{x}{1+x}) x^{-\frac{x}{(1+x)}} \text{ for } x > 0.$$

PROOF. First note that it is straightforward to show that  $\mathbb{E}[e^{-L}] = \mathbb{E}[(1+M_1(\zeta))^{-1}] = e^{-1}$ . This fact also establishes the existence of  $L$ . Now we see that  $G_1/(G_1+E) \stackrel{d}{=} U_{[0,1]}$ . The result then follows by applying statement [(ii)] of Proposition 3.2 to (6)  $\square$

In view of the remarks in Section 3.0.1, we now give a description of the laws of the subordinators associated with the two random variables above.

**Proposition 3.4** *Suppose the  $Z(t)$  is a subordinator where  $Z(1)$  is  $\text{GGC}(1, U[0, 1])$ , then for  $0 < t < 1$ ,  $Z(t) \stackrel{d}{=} G_1 M_1(U^{(t)}[0, 1])$ , where  $M_1(U^{(t)}[0, 1])$  has density,*

$$\frac{e^t}{\pi} \sin(\pi t(1-y)) y^{t(1-y)-1} (1-y)^{-t(1-y)} \text{ for } 0 < y < 1.$$

*If  $Z(t)$  is such that  $Z(1)$  is  $\text{GGC}(1, \zeta)$  then for  $0 < t < 1$ ,  $Z(t) \stackrel{d}{=} G_1 M_1(\zeta^{(t)})$ , where  $M_1(\zeta^{(t)})$  has density,*

$$\frac{1}{\pi} \sin\left(\pi \frac{t}{1+x}\right) x^{\frac{t}{(1+x)}-1} \text{ for } x > 0.$$

PROOF. This now follows from Theorem 3.1, Proposition 3.3 and (6).  $\square$

Once we have the density in  $M_1(\zeta)$  we can then extend the result for  $M_1(U[0, 1])$  to that of  $M_1(O_{1/2,p}^{br})$ , where  $O_{1/2,p}^{br}$  denotes the distribution of the random variable

$$A_{1/2,p}^{br} = \int_0^1 \mathbb{I}(B_p^{(1/2,1/2)}(s) > 0) ds \stackrel{d}{=} \frac{p^2 G_1}{p^2 G_1 + q^2 E}.$$

In the notation above  $B_p^{(1/2,1/2)}(s)$  denotes a  $p$ -skew Brownian motion. Hence  $A_{1/2,p}^{br}$  denotes the time spent positive by this process up till time 1. The following result is otherwise not obvious.

**Proposition 3.5** *Define  $A_{1/2,p}^{br} \stackrel{d}{=} p^2 G_1 / [p^2 G_1 + q^2 E]$ . The random variable  $A_{1/2,p}^{br}$  is equivalent in distribution to the time spent positive of a  $p$ -skew Brownian bridge having density  $O_{1/2,p}^{br}(dy)/dy = p^2 q^2 / [p^2(1-y) + q^2 y]$ . Now let  $L_p$  denote a  $\text{GGC}(1, O_{1/2,p}^{br})$  random variable. Then equivalently the Lévy measure of  $L_p$  is given by*

$$s^{-1} \mathbb{E}[e^{-s/A_{1/2,p}^{br}}]$$

Furthermore  $L_p \stackrel{d}{=} G_1 M_1(O_{1/2,p}^{br})$ , where  $M_1(O_{1/2,p}^{br})$  has density,

$$f_{M_1(O_{1/2,p}^{br})}(y) = \frac{\kappa_p}{\pi} \sin\left(\frac{\pi q^2 y}{p^2(1-y) + q^2 y}\right) y^{-\frac{q^2 y}{p^2(1-y) + q^2 y}} (1-y)^{-\frac{p^2(1-y)}{p^2(1-y) + q^2 y}} \text{ for } 0 < y < 1.$$

Where for  $c = p^2/q^2$

$$1/\kappa_p = c \mathbb{E}[(1 + cM(\zeta))^{-1}] = c e^{-\int_0^\infty \frac{\log(1+cx)}{(1+x)^2} dx}$$

REMARK 8. The specific densities in Proposition 3.3 and 3.5 yield the not immediately obvious identities.

$$\Phi_1(x) = -\int_0^\infty \frac{\log(|x-y|)}{(1+y)^2} dy = -\frac{x}{1+x} \log(x)$$

and, more so,

$$\Phi_2(x) = -\int_0^1 \frac{\log(|x-y|) p^2 q^2}{[p^2(1-y) + q^2 y]} dy = \log\left(\kappa_p y^{-\frac{q^2 y}{p^2(1-y) + q^2 y}} (1-y)^{-\frac{p^2(1-y)}{p^2(1-y) + q^2 y}}\right)$$

where these are appropriate versions of  $\Phi$ .



### 3.3 Reconciling some results of Cifarelli and Mellili

To further illustrate our point we show how to reconcile two apparently unrelated results given in Cifarelli and Mellili (2000). Let  $\Lambda_{1/2,1/2}$  denote the distribution of the arcsine law, that is a  $B_{1/2,1/2}$  random variable. Cifarelli and Mellili (2000, p.1394-195) show that for all  $\theta > 0$   $M_\theta(\Lambda_{1/2,1/2}) \stackrel{d}{=} B_{\theta+1/2,\theta+1/2}$ . Now define the probability density

$$\varrho_{1/2}(x) = \frac{1}{\pi} x^{-1/2} (1+x)^{-1}.$$

Cifarelli and Mellili then show that for  $\theta \geq 1$   $M_\theta(\varrho_{1/2})$  has the density proportional to

$$x^{\theta-1/2} (1+x)^{-(\theta+1)}.$$

Hjort and Ongaro (2005) recently extend this result for all  $\theta > 0$  and also note the normalizing constant appearing in Cifarelli and Mellili (2001) is incorrect. Here however we note that if  $X$  has density  $\varrho_{1/2}$  then  $X \stackrel{d}{=} G_{1/2}/G'_{1/2}$  where  $G_{1/2}$  and  $G'_{1/2}$  are independent and identically distributed gamma random variables. Now using the known fact that

$$B_{1/2,1/2} \stackrel{d}{=} \frac{G_{1/2}/G'_{1/2}}{G_{1/2}/G'_{1/2} + 1},$$

we see that  $B_{1/2,1/2}$  is a special case of  $A_1$  in Proposition 3.2. It is now evident that one could use the result  $M_\theta(\Lambda_{1/2,1/2}) \stackrel{d}{=} B_{\theta+1/2,\theta+1/2}$ , coupled with statement [(ii)] of Proposition 3.2, to easily obtain the density of  $M_\theta(\varrho_{1/2})$  for all  $\theta > 0$ . Similar to section 3.2 one could then use the density of  $M_\theta(\varrho_{1/2})$  to obtain results for mean functionals based on the law of  $p^2 G_{1/2}/[p^2 G_{1/2} + q^2 G'_{1/2}]$ . We will encounter this class of models again in the next coming sections and see how they arise as laws of occupations times of Brownian bridge and related models.

## 4 A tilted positive Linnik process

In this section we present details of a subordinator,  $Z$ , which is a FGGC such that  $Z(1)$  has the Laplace transform,

$$(7) \quad \left[ \frac{1}{1+c^\alpha} (1+(c+b\lambda)^\alpha) \right]^{-\theta}$$

for parameters  $0 < \alpha < 1$ ,  $\theta > 0$ ,  $c \geq 0$  and  $b > 0$ . The first important thing to note is that  $Z(t)$  is equivalent in distribution to that of  $Z(1)$  with the parameter  $\theta$  replaced by  $\theta t$ . This as we shall show more specifically, yields the desirable property of being able to explicitly identify the distribution of  $Z(t)$  for all  $t$ . In this generality we believe that this process has not been studied in any detail, but setting  $c = 0$ , we see that the Laplace transform becomes

$$(1+b^\alpha \lambda^\alpha)^{-\theta}.$$

It is easy to verify that this corresponds to the random variable defined as

$$bL_{\alpha,\theta} := bG_\theta^{1/\alpha} S_\alpha.$$

The random variable,  $L_{\alpha,\theta}$ , is known in the literature[see for instance Bondesson (1992, p.38) and Devroye (1990, 1996)] and is sometimes called a positive Linnik process and has a host of interesting properties and distributional representations. Now let us introduce the random variable which will play a key role throughout the remainder of the paper. Let  $X_\alpha \stackrel{d}{=} S_\alpha/S'_\alpha$  denote the random variable having density

$$(8) \quad \varrho_\alpha(y) = \frac{\sin(\pi\alpha)}{\pi} \frac{y^{\alpha-1}}{y^{2\alpha} + 2y^\alpha \cos(\pi\alpha) + 1} \text{ for } y > 0.$$

Then the Lévy exponent associated with  $L_{\alpha,\theta}$  is

$$\tilde{\psi}_{\alpha,\theta}(\omega) := \theta \ln(1 + \omega^\alpha) = \int_0^\infty (1 - e^{-\lambda s}) l_{\theta,\alpha}(s) ds$$

where

$$l_{\theta,\alpha}(s) = \frac{\alpha\theta}{s} \phi_\alpha(s^\alpha) = \frac{\alpha\theta}{s} \mathbb{E}[e^{-sX_\alpha}] = \alpha\theta s^{-1} \mathbb{E}[e^{-s/X_\alpha}]$$

is the Lévy density of the Linnik process. Specifically,

$$\phi_\alpha(q) = \mathbb{E}[e^{-qS_\alpha^{-\alpha}}] = \sum_{k=0}^{\infty} \frac{1}{\Gamma(1+k\alpha)} (-q)^k = \mathbb{E}[e^{-q^{1/\alpha}X_\alpha}]$$

is the Mittag Leffler function, which equates with the Laplace transform of  $S_\alpha^{-\alpha}$ . The equivalences involving  $X_\alpha$  are probably not that well known but serve to identify the density of  $X_\alpha$  (8) as the Thorin measure,  $\nu$  of  $L_{\alpha,\theta}$ . This Thorin measure is identified by Bondesson ((1992, p. 38) and confirms the fact that  $L_{\alpha,\theta}$  is an FGFC. That is,

$$\tilde{\psi}_{\alpha,\theta}(\omega) := \theta\alpha \int_0^\infty (1 - e^{-\omega s}) s^{-1} \int_0^\infty e^{-s/x} \varrho_\alpha(x) dx ds.$$

One of the things will be looking for are alternative representations of the distribution of  $L_{\alpha,\theta}$ . For instance it is known from Devroye (1996), that when  $\theta = 1$  one has  $L_{\alpha,1} = G_1 X_\alpha$ , where  $G_1$  is exponential (1) and  $X_\alpha$  has density (8). Summarizing, we see that  $L_{\alpha,\theta}$  is a GGC( $\alpha\theta, \varrho_\alpha$ ) and (7) corresponds to the class GGC<sup>(b,c)</sup>( $\alpha\theta, \varrho_\alpha$ ) We will establish various distributional equivalences which then lead to results for Dirichlet mean functionals and Bessel occupation times. First we describe some more pertinent features of  $X_\alpha$ .

**Proposition 4.1** *Let  $X_\alpha \stackrel{d}{=} S_\alpha/S'_\alpha$ , having density (8). Then,*

(i) *The cdf of  $X_\alpha$  can be represented explicitly as*

$$(9) \quad F_{X_\alpha}(x) = 1 - \frac{1}{\pi\alpha} \cot^{-1} \left( \cot(\pi\alpha) + \frac{x^\alpha}{\sin(\pi\alpha)} \right)$$

(ii) *Its inverse is given by*

$$(10) \quad F_{X_\alpha}^{-1}(y) = \left[ \frac{\sin(\pi\alpha(y))}{\sin(\pi\alpha(1-y))} \right]^{1/\alpha}$$

(iii) *The equations (9) and (10) yield the quite useful identity,*

$$(11) \quad \sin(\pi\alpha(1 - F_{X_\alpha}(y))) = y^{-\alpha} \sin(\pi\alpha F_{X_\alpha}(y)) = \frac{\sin(\pi\alpha)}{[y^{2\alpha} + 2y^\alpha \cos(\pi\alpha) + 1]^{1/2}}$$

**PROOF.** This derivation of the cdf is influenced by arguments in Fujita and Yor (2006) where it becomes clear that it is easier to work with the density of  $[X_\alpha]^{1/\alpha}$ . The cdf of this quantity is easily obtained. Statements [(ii)] and [(iii)] then become apparent.  $\square$

**REMARK 9.** There are several things interesting about the alternative representation of the Lévy measure involving  $X_\alpha$ . First, from Chaumont and Yor (2003, sec. 4.19 and 4.21), the distributional equivalence can also be seen to arise from the fact that  $G_1^{1/\alpha} \stackrel{d}{=} G_1/S'_\alpha$ , where  $S'_\alpha$  is a stable random variable independent of  $S_\alpha$ , and the fact that  $X_\alpha \stackrel{d}{=} S_\alpha/S'_\alpha$ . The density of  $X_\alpha$  can be traced back to the work of Lamperti (1958) but arises later elsewhere. One notes also that the density for  $X_\alpha$  has a simple form as compared with a stable law of index  $0 < \alpha < 1$ .

REMARK 10. One notes that the random variable  $L_{\alpha,\theta}$  is conditionally a stable random variable and hence is heavy-tailed. So this limits its practical applicability to a variety of problems. A natural way to create a random variable which has moments is to exponential tilt the density of  $L_{\alpha,\theta}$ . This is precisely how the Laplace transform, (7), comes about. It is easy to see that,

$$\mathbb{E}[e^{-\lambda Z(1)}] = \frac{\mathbb{E}[e^{-(b\lambda+c)L_{\alpha,\theta}}]}{\mathbb{E}[e^{-cL_{\alpha,\theta}}]}$$

In other words the density of  $Z(1)$  is given by

$$f_{Z(1)}(x) = \frac{1}{b}(1+c^\alpha)e^{-xc/b}f_{L_{\alpha,\theta}}(x/b) = \frac{e^{-\frac{p}{b}(p/q)^{1/\alpha}}}{b(1-p)}f_{L_{\alpha,\theta}}(x/b).$$

#### 4.1 Connection to occupation times of Bessel bridges and $P_{\alpha,\theta}(C)$

Let  $J_1 \geq J_2 \geq \dots$  denote the ranked jump sizes of an  $(\alpha)$ -stable subordinator such that  $\sum_{k=1}^{\infty} J_k = T_{\alpha,0}$  is a unilateral  $(\alpha)$ -stable random variable. Recall from Pitman and Yor (1997a) that the sequence of ranked probabilities  $(P_k = J_k/T_{\alpha,0})$  is said to have a two-parameter Poisson Dirichlet with specification  $PD(\alpha, 0)$ . Furthermore, for  $\alpha\theta > 0$ , one obtains the  $PD(\alpha, \alpha\theta)$  law of  $(P_k)$  by mixing over the *conditional* law of  $(P_k)|T_{\alpha,0} = t$  with respect to the distribution of the random variable, say  $T_{\alpha,\alpha\theta}$ , which has density  $\mathbb{P}(T_{\alpha,\alpha\theta} \in dt) \propto t^{-\alpha\theta}\mathbb{P}(T_{\alpha,0} \in dt)$ . One may then introduce, independent of  $(P_k)$ , a sequence of iid random variables  $(Z_k)$  having some common non-atomic law. Then as in Pitman (1996), see also Ishwaran and James (2001), analogous to the Dirichlet process, one may define the class of  $PD(\alpha, \alpha\theta)$  random probability measures as  $P_{\alpha,\alpha\theta}(\cdot) = \sum_{k=1}^{\infty} P_k \delta_{Z_k}(\cdot)$ .

Now it will be shown that the exponential tilting operation to obtain (7) reveals a strong, albeit initially unexpected, connection to work of Pitman and Yor (1997b) and Barlow, Pitman and Yor (1989), on occupation time models for skew Bessel bridges and more generally laws of  $P_{\alpha,\alpha\theta}(C)$  for some set  $C$  such that  $\mathbb{E}[P_{\alpha,\theta}(C)] = p$ .

This is seen for the case where  $c \neq 0$ , which allows one to to rewrite (7) as,

$$(12) \quad \frac{1}{(q + p(1 + \frac{b}{c}\lambda)^\alpha)^\theta}$$

for  $p = c^\alpha/1 + c^\alpha$  and  $q = 1 - p$ . We recognize that for  $b = c$ , this equates with the Cauchy-Stieltjes transform of order  $\alpha\theta$  of a two parameter  $(\alpha, \alpha\theta)$  Poisson Dirichlet random probability measure evaluated at some set  $C$ , denoted as  $P_{\alpha,\alpha\theta}(C)$ , where  $\mathbb{E}[P_{\alpha,\alpha\theta}(C)] = p$ . That is,

$$\mathbb{E}[(1 + \lambda P_{\alpha,\alpha\theta}(C))^{-\theta\alpha}] = \frac{1}{(q + p(1 + \lambda)^\alpha)^\theta}.$$

Furthermore, when  $\theta = 1$ , this is the Cauchy-Stieltjes transform of order  $\alpha$  of the time spent positive up to time 1 of a  $p$ -skew Bessel bridge of dimension  $2 - 2\alpha$ , as can be seen from Pitman and Yor (1997b, eq(75)). This random variable can be represented as,

$$A_{\alpha,p}^{br} \stackrel{d}{=} P_{\alpha,\alpha}(C) = \int_0^1 \mathbb{I}(B_p^{(\alpha,\alpha)}(s) > 0) ds$$

where  $B_p^{(\alpha,\alpha)}$  is the corresponding  $p$ -skew Bessel bridge. We also note another important connection to Lamperti (1958) and Barlow, Pitman and Yor (1989), Pitman and Yor (1997b). Noting that  $c = (p/q)^{1/\alpha}$ , the random variable

$$(13) \quad A_{\alpha,p} \stackrel{d}{=} P_{\alpha,0}(C) \stackrel{d}{=} \frac{cX_\alpha}{1 + cX_\alpha} \stackrel{d}{=} \frac{p^{1/\alpha}S'_\alpha}{p^{1/\alpha}S'_\alpha + q^{1/\alpha}S_\alpha} \stackrel{d}{=} \int_0^1 \mathbb{I}(B_p^{(\alpha)}(s) > 0) ds$$

equates with the time spent positive by  $B_p^{(\alpha)}$ , now a  $p$ -skew Bessel process of dimension  $2 - 2\alpha$  with skewness parameter  $p$ , up to time 1. When  $p = 1/2$ , one obtains the usual Bessel processes. Lamperti (1958) shows that the density of  $A_{\alpha,p}$  is

$$(14) \quad \Lambda_{\alpha,p}(dx)/dx = \frac{p q \sin(\alpha\pi) x^{\alpha-1} (1-x)^{\alpha-1}}{\pi [q^2 x^{2\alpha} + p^2 (1-x)^{2\alpha} + 2pq x^\alpha (1-x)^\alpha \cos(\alpha\pi)]},$$

We see that setting  $p = 1/2, \alpha = 1/2$ , yields Lévy's (1939) famous result that the time spent positive by Brownian motion up to time 1 has the Arcsine distribution. That is,  $A_{1/2,1/2}$  is BETA(1/2, 1/2). Now interestingly from (9) and (10) we obtain a closed form expression for the cdf and quantile function of  $A_{\alpha,p}$ , as

$$(15) \quad F_{A_{\alpha,p}}(y) = 1 - \frac{1}{\pi\alpha} \cot^{-1} \left( \cot(\pi\alpha) + \frac{qy^\alpha}{p(1-y)^\alpha \sin(\pi\alpha)} \right)$$

and its inverse given by

$$(16) \quad F_{A_{\alpha,p}}^{-1}(y) = \frac{\left[ \frac{p \sin(\pi\alpha(y))}{q \sin(\pi\alpha(1-y))} \right]^{1/\alpha}}{1 + \left[ \frac{p \sin(\pi\alpha(y))}{q \sin(\pi\alpha(1-y))} \right]^{1/\alpha}} = \frac{F_{X_\alpha}^{-1}(y)}{1 + F_{X_\alpha}^{-1}(y)}$$

REMARK 11. Hereafter, for  $c \neq 0$ , we shall set  $c^\alpha = p/q$ . Note that if  $B(s)$  denotes in a generic sense a Bessel process or bridge, then the interpretation of  $p$  is that the  $p$ -skewed version of  $B(s)$  has the property that

$$p = \mathbb{P}(B(s) > 0).$$

Now we relate the more general class of PD( $\alpha, \alpha\theta$ ) models to occupation laws of processes and their accompanying local times. Noting Pitman and Yor (1992, p. 332) let  $(\ell_t^{(\alpha)}, t \geq 0)$  denote the right continuous local time of a Bessel process, and let  $S_\alpha(t) \stackrel{d}{=} S_\alpha t^{1/\alpha}$  denote an  $(\alpha)$ -stable subordinator, which satisfies the identities in law,

$$(17) \quad S_\alpha(s) = \inf\{t : \ell_t^{(\alpha)} > s\} \text{ and } \frac{\ell_t^{(\alpha)}}{t^\alpha} \stackrel{d}{=} \frac{s}{(S_\alpha(s))^\alpha} \stackrel{d}{=} \frac{1}{(S_\alpha)^\alpha}.$$

For our purposes, we may set  $S_\alpha(1) = T_{\alpha,0}$ , and furthermore  $[\ell_1^{(\alpha)}]^{1/\alpha} \stackrel{d}{=} 1/T_{\alpha,0}$ . Now using the scaling property (17) one may construct local times  $\{\ell_t^{(\alpha,\alpha\theta)} : t \geq 0\}$  with laws specified by  $\mathbb{P}([\ell_1^{(\alpha,\alpha\theta)}]^{1/\alpha} \in ds) = \mathbb{P}(1/T_{\alpha,\alpha\theta} \in ds)$ . As a special case,  $\ell^{(\alpha,\alpha)}$  denotes the local time of a Bessel bridge, say  $B^{(\alpha,\alpha)}$ . Associated with  $\{\ell_t^{(\alpha,\alpha\theta)} : t \geq 0\}$  are what we shall call, *generalized Bessel bridges* say  $B^{(\alpha,\alpha\theta)}(t)$  with law specified by  $\int_0^\infty \mathbb{P}(B^{(\alpha)}(t) | \ell_1^{(\alpha)} = s) \mathbb{P}(\ell_1^{(\alpha,\alpha\theta)} \in ds)$ . Such processes may be found in Definition 3.14 of Perman, Pitman and Yor (1992). Now letting  $B_p^{(\alpha,\alpha\theta)}$  denote a  $p$ -skewed version of such processes we define their times spent positive up to time 1 as,  $A_{\alpha,p}^{(\alpha,\alpha\theta)}$  satisfying,

$$A_{\alpha,p}^{(\alpha,\alpha\theta)} \stackrel{d}{=} P_{\alpha,\alpha\theta}(C)$$

Note that  $B_{1/2}^{(\alpha,\alpha\theta)} := B^{(\alpha,\alpha\theta)}$ .

REMARK 12. The representation of the occupation time of a skew Bessel process in (13) was given by Barlow, Pitman and Yor (1989). It will play a fundamental role in our understanding of the construction of randomized versions and related matters. In effect this boils down to the interpretation of  $c$  as  $c^\alpha = p/q$

REMARK 13. In this remark we demonstrate how the positive Linnik may be interpreted as the distribution of a time changed occupation time. It is known, see Barlow, Pitman and Yor (1989) or section 4 of Pitman and Yor (1992), that for  $S_\alpha(t)$  an inverse local time,

$$A_{\alpha,p}(S_\alpha(t)) = \int_0^{S_\alpha(t)} \mathbb{I}(B_{s,p}^{(\alpha)} > 0) ds \stackrel{d}{=} S_\alpha(p)t^{1/\alpha} \stackrel{d}{=} S_\alpha(pt) \stackrel{d}{=} p^{1/\alpha}t^{1/\alpha}S_\alpha$$

is an  $(\alpha)$ -stable subordinator. Hence letting  $(\Gamma_\theta(t), t \geq 0)$  denote an independent gamma subordinator satisfying for each fixed  $t$ ,  $\Gamma_\theta(t) \stackrel{d}{=} G_{\theta t}$ , it follows that

$$A_{\alpha,p}(S_\alpha(\Gamma_\theta(t))) \stackrel{d}{=} S_\alpha(p\Gamma_\theta(t)) \stackrel{d}{=} p^{1/\alpha}G_{\theta t}^{1/\alpha}S_\alpha = p^{1/\alpha}S_\alpha(\Gamma_\theta(t)).$$

REMARK 14. The random variables  $T_{\alpha,\alpha\theta}^{-\alpha}$  are the  $\alpha$ -diversity of the  $\text{PD}(\alpha, \alpha\theta)$  exchangeable partitions as described in Pitman (2003, Proposition 13). Furthermore it may be read from Perman, Pitman and Yor (1992, p. 31) and Pitman (2006) that for  $\theta \geq 1$   $T_{\alpha,\theta\alpha} \stackrel{d}{=} T_{\alpha,\alpha\theta-\alpha}U_{\theta\alpha,1-\alpha}$ , where  $U_{\theta\alpha,1-\alpha}$  is independent of  $T_{\alpha,\theta\alpha}$  but not of  $T_{\alpha,\alpha\theta-\alpha}$ , and marginally  $U_{\theta\alpha,1-\alpha} \stackrel{d}{=} B_{\alpha\theta,1-\alpha}$ . We will sometimes refer to all the relevant local times and occupations times, already defined, as  $\text{PD}(\alpha, \alpha\theta)$  processes.

## 4.2 Distributional results for the Linnik class and local times

We first establish various distributional identities related to the positive Linnik random variable. Importantly, we will show that random variables based on the  $\text{PD}(\alpha, \alpha\theta)$  models are Dirichlet mean functionals.

REMARK 15. Throughout we will be using the fact that if  $X$  is a gamma or  $(\alpha)$ -stable random variable, then the independent random variables  $X, Y, Z$  satisfying  $XY \stackrel{d}{=} XZ$  imply that  $Y \stackrel{d}{=} Z$ . For precise conditions see Chaumont and Yor (2003, sec. 1.12 and 1.13).

**Theorem 4.1** *Let  $L_{\alpha,\theta} = G_\theta^{1/\alpha}S_\alpha$  denote a generalized Linnik random variable i.e. a  $\text{GGC}(\alpha\theta, \varrho_\alpha)$ . Then, we have the distributional equivalences*

(i) *For all  $\theta > 0$ ,  $L_{\alpha,\theta} \stackrel{d}{=} G_{\theta\alpha}S_\alpha/T_{\alpha,\alpha\theta} \stackrel{d}{=} G_{\theta\alpha}M_{\alpha\theta}(\varrho_\alpha)$ , which implies*

$$S_\alpha/T_{\alpha,\alpha\theta} \stackrel{d}{=} M_{\alpha\theta}(\varrho_\alpha) \stackrel{d}{=} [\ell_{S_\alpha}^{(\alpha,\alpha\theta)}]^{1/\alpha}$$

(ii) *In particular when  $\theta = 1$ ,  $L_{\alpha,1} \stackrel{d}{=} G_1X_\alpha \stackrel{d}{=} G_1B_{\alpha,1-\alpha}M_\alpha(\varrho_\alpha)$ . Hence*

$$X_\alpha \stackrel{d}{=} B_{\alpha,1-\alpha}M_\alpha(\varrho_\alpha) \stackrel{d}{=} M_1(\varrho_\alpha^{(\alpha)})$$

(iii) *For  $0 < \theta < 1$ ,*

$$L_{\alpha,\theta} \stackrel{d}{=} B_{\theta,1-\theta}^{1/\alpha}G_1X_\alpha \stackrel{d}{=} G_1B_{\theta\alpha,1-\theta\alpha}\frac{S_\alpha}{T_{\alpha,\alpha\theta}} \stackrel{d}{=} G_{\theta\alpha+1-\alpha}B_{\alpha\theta,1-\alpha}\frac{S_\alpha}{T_{\alpha,\alpha\theta}}$$

(iv) *Statement (i) implies  $G_\theta^{1/\alpha} \stackrel{d}{=} G_{\alpha\theta}/T_{\alpha,\alpha\theta}$ , for  $\theta > 0$ . For  $0 < \theta < 1$ ,*

$$G_\theta^{1/\alpha} \stackrel{d}{=} \frac{G_{\alpha\theta}}{T_{\alpha,\alpha\theta}} \stackrel{d}{=} \frac{G_1B_{\theta,1-\theta}^{1/\alpha}}{S_\alpha} \stackrel{d}{=} \frac{G_{\alpha\theta+1-\alpha}}{T_{\alpha,\alpha\theta}}B_{\alpha\theta,1-\alpha}$$

When  $\theta = 1$ , then

$$G_1^{1/\alpha} \stackrel{d}{=} \frac{G_\alpha}{S_{\alpha,\alpha}} \stackrel{d}{=} \frac{G_1}{S_\alpha}$$

PROOF. The first equivalence in statement (i) is perhaps the most non-obvious. The result is obtained by manipulating the density representation of  $G_\theta^{1/\alpha} S_\alpha$  as follows. Let  $f_\alpha$  denote the density of an  $(\alpha)$ -stable random variable. Now note that the density of  $L_{\alpha,\theta}$  is obviously expressible as,

$$f_{L_{\alpha,\theta}}(y) = Cy^{\alpha\theta-1} \int_0^\infty s^{-\theta\alpha} e^{-(y/s)^\alpha} f_\alpha(s) ds = Cy^{\alpha\theta-1} \int_0^\infty s^{-\theta\alpha} \mathbb{E}[e^{-(y/s)S_\alpha}] f_\alpha(s) ds.$$

for some constant C. Now it remains to write

$$\mathbb{E}[e^{-(y/s)S_\alpha}] = \int_0^\infty e^{-vy/s} f_\alpha(v) dv.$$

The result is then obtained by algebraic manipulations. The second equivalence is immediate since  $L_{\alpha,\theta}$  is  $\text{GGC}(\alpha\theta, \varrho_\alpha)$ . The remaining statements are straightforward applications of beta-gamma calculus and independence.  $\square$

REMARK 16. Statement [(iv)] generalizes the known case of  $G_1^{1/\alpha} \stackrel{d}{=} G_1/S_\alpha$ . It is interesting to note that this special case can be interpreted in terms of stochastic processes, as,

$$\ell_{G_1}^{(\alpha)} \stackrel{d}{=} G_1^\alpha \frac{1}{(S_\alpha)^\alpha} \stackrel{d}{=} G_1$$

where  $\ell_{G_1}^{(\alpha)}$  is the local time of a Bessel process with dimension  $2 - 2\alpha$  considered at the independent exponential time  $G_1$ . This description, and further references, may be found in Chaumont and Yor (2003, p. 114). It is also known from Barlow, Pitman and Yor (1989) that the local time for the Bessel bridges,  $\ell_t^{(\alpha,\alpha)}$  evaluated at  $G_\alpha$ , satisfies  $\ell_{G_\alpha}^{(\alpha,\alpha)} \stackrel{d}{=} \ell_{G_1}^{(\alpha)}$ . Now using the scaling property in (17) it follows from statement [(iv)] that,

$$\ell_{(G_{\theta\alpha})}^{(\alpha,\alpha\theta)} \stackrel{d}{=} \frac{(G_{\theta\alpha})^\alpha}{(T_{\alpha,\alpha\theta})^\alpha} \stackrel{d}{=} G_\theta.$$

See section 5 for an extension of this idea to more general random times.

Theorem 4.1 establishes the distributional results,

$$X_\alpha \stackrel{d}{=} B_{\alpha,1-\alpha} M_\alpha(\varrho_\alpha) \stackrel{d}{=} M_1(\varrho_\alpha^{(\alpha)}) = [(\ell_{B_{\alpha,1-\alpha}}^{(\alpha,\alpha)})]^{1/\alpha} S_\alpha \stackrel{d}{=} [(\ell_{S_\alpha}^{(\alpha)})]^{1/\alpha}$$

where  $(\ell_{S_\alpha}^{(\alpha)})$  denotes the local time of a Bessel process evaluated at an independent unilateral  $(\alpha)$ -stable random variable,  $S_\alpha$ . The fact that it is representable as a Dirichlet mean functional,  $M_1(\varrho_\alpha^{(\alpha)})$  coupled with the explicit density of  $X_\alpha$ , leads to an important identity below, which plays a fundamental role in obtaining explicit densities throughout the rest of this work.

**Proposition 4.2** Define  $\mathcal{S}_\alpha(x) = \int_0^\infty \log(|x-y|) \varrho_\alpha(dy) = \mathbb{E}[\log(|x-X_\alpha|)]$ . Then for  $0 < \alpha < 1$ ,

$$\mathcal{S}_\alpha(x) = \frac{1}{2\alpha} \log(x^{2\alpha} + 2x^\alpha \cos(\alpha\pi) + 1).$$

Note that  $\mathcal{S}_\alpha(x)$  is a special case of  $\Phi(x)$ .

PROOF. From Theorem 3.1. it follows that the density of  $X_\alpha \stackrel{d}{=} M_1(\varrho_\alpha^{(\alpha)})$  satisfies the equivalence,

$$\varrho_\alpha(x) = \frac{1}{\pi} \sin(\pi\alpha[1 - F_{X_\alpha}(x)]) e^{-\mathcal{S}_\alpha(x)} x^{\alpha-1}.$$

Solving this expressions for  $\mathcal{S}_\alpha(x)$ , and applying the identity in (11) concludes the result.  $\square$

With this we obtain explicit expressions for the cdf and density of  $M_{\alpha\theta}(\varrho_\alpha)$  as follows;

**Theorem 4.2** From Theorem 4.1,  $M_{\alpha\theta}(\varrho_\alpha) \stackrel{d}{=} [\ell_{S_\alpha}^{(\alpha, \alpha\theta)}]^{1/\alpha} \stackrel{d}{=} S_\alpha/T_{\alpha, \alpha\theta}$ . The form of the cdf for  $M_{\alpha\theta}(\varrho_\alpha)$  for all  $\alpha\theta > 0$ , is given by (1), with  $\theta := \alpha\theta$ , and

$$\Delta_{\alpha\theta}(x|\varrho_\alpha) = \frac{1}{\pi} \frac{\sin(\pi\theta\alpha F_{X_\alpha}(x))}{[x^{2\alpha} + 2x^\alpha \cos(\alpha\pi) + 1]^{\theta/2}}$$

where  $F_{X_\alpha}$  is given in (9). Furthermore, a general expression for the density is obtained from (3) with  $\theta := \alpha\theta$  and

$$(18) \quad d_{\alpha\theta}(x|\varrho_\alpha) = \frac{\alpha\theta x^{\alpha-1}}{\pi} \frac{[\sin(\pi\alpha[1 - \theta F_{X_\alpha}(x)]) - x^\alpha \sin(\pi\theta\alpha F_{X_\alpha}(x))]}{[x^{2\alpha} + 2x^\alpha \cos(\alpha\pi) + 1]^{\theta/2+1}}.$$

□

PROOF. The bulk of the result is a straightforward application of Proposition 4.2 combined with the explicit forms of the densities in Section 3. The expression in (18) follows by, noting the explicit form of the density  $\varrho_\alpha$ , and applying the identity  $\sin(w - z) = \sin(w)\cos(z) - \sin(z)\cos(w)$ , with  $w = \pi\alpha$  and  $z = \pi\theta\alpha F_{X_\alpha}(x)$  □

From this we obtain the most explicit case of  $\alpha\theta = 1$ , and the important case where  $\alpha\theta = \alpha$ , related to the local time of a Bessel Bridge evaluated at an independent stable time.

**Corollary 4.1** Consider the random variables in Theorem 4.2 then, for  $\theta = 1$ , corresponding to the Bessel bridge, the random variable  $M_\alpha(\varrho_\alpha) \stackrel{d}{=} [\ell_{S_\alpha}^{(\alpha, \alpha)}]^{1/\alpha} \stackrel{d}{=} S_\alpha/T_{\alpha, \alpha}$  has density determined by (3) where  $d_\alpha(x|\varrho_\alpha)$  simplifies to,

$$d_\alpha(x|\varrho_\alpha) = \frac{\alpha x^{\alpha-1}}{\pi} \frac{(1 - x^{2\alpha}) \sin(\pi\alpha)}{[x^{2\alpha} + 2x^\alpha \cos(\alpha\pi) + 1]^2} \text{ for } x > 0.$$

When  $\theta = 1/\alpha$ , the density of  $M_1(\varrho_\alpha) \stackrel{d}{=} [\ell_{S_\alpha}^{(\alpha, 1)}]^{1/\alpha} \stackrel{d}{=} S_\alpha/T_{\alpha, 1}$  is given by

$$f_{M_1}(x|\varrho_\alpha) = \Delta_1(x|\varrho_\alpha) = \frac{1}{\pi} \frac{\sin(\pi F_{X_\alpha}(x))}{[x^{2\alpha} + 2x^\alpha \cos(\alpha\pi) + 1]^{1/2\alpha}}.$$

PROOF. Apply (11) to obtain the expression for  $d_\alpha(x|\varrho_\alpha)$ . □

REMARK 17. It is evident from Theorem 3.1 and Proposition 4.2, that we can also obtain similar types of expressions, as in Theorem 4.2, for the densities of  $M_\theta(\varrho_\alpha^{(\alpha)}) \stackrel{d}{=} B_{\alpha\theta, \theta(1-\alpha)} M_{\alpha\theta}(\varrho_\alpha)$ , for all  $\theta > 0$ . For brevity we do not give that here as it can be easily deduced. Instead we concentrate on the interesting class of  $M_1(\varrho_\alpha^{(\alpha\theta)}) \stackrel{d}{=} B_{\alpha\theta, 1-\alpha\theta} M_{\alpha\theta}(\varrho_\alpha)$  for  $0 < \alpha\theta < 1$ . These have the densities of non-integral form.

**Proposition 4.3** For  $0 < \alpha\theta < 1$ ,

$$M_1(\varrho_\alpha^{(\alpha\theta)}) \stackrel{d}{=} B_{\alpha\theta, 1-\alpha\theta} M_{\alpha\theta}(\varrho_\alpha) \stackrel{d}{=} [(\ell_{B_{\alpha\theta, 1-\alpha\theta}}^{(\alpha, \alpha\theta)})]^{1/\alpha} S_\alpha$$

and has density

$$(19) \quad \frac{1}{\pi} \frac{\sin(\theta C_\alpha(x)) x^{\alpha\theta-1}}{[x^{2\alpha} + 2x^\alpha \cos(\alpha\pi) + 1]^{\theta/2}}$$

where  $C_\alpha(x) = \cot^{-1} \left( \cot(\pi\alpha) + \frac{x^\alpha}{\sin(\pi\alpha)} \right)$ . The density equates with  $\varrho_\alpha$  when  $\theta = 1$ . For  $0 < \theta < 1$  (19) equates with the density of  $B_{\theta, 1-\theta}^{1/\alpha} X_\alpha$ .

PROOF. This is a consequence of Theorems 3.1 and 4.1  $\square$

When  $\alpha = 1/k$ , we obtain representations in terms of products of gamma random variables.

**Proposition 4.4** *Suppose that  $\alpha = 1/k$  for some integer  $k = 2, 3, \dots$ . Then since for  $\theta > 0$ ,  $L_{1/k, \theta} \stackrel{d}{=} S_{1/k} G_{\theta}^k \stackrel{d}{=} G_{\theta/k} M_{\theta/k}(\varrho_{1/k})$ , one deduces that,*

$$M_{\theta/k}(\varrho_{1/k}) = \frac{S_{1/k}}{T_{1/k, \theta/k}} \stackrel{d}{=} k^k S_{1/k} \prod_{j=1}^{k-1} G_{\frac{\theta+j}{k}}$$

This implies  $T_{1/k, \theta/k}$ , is such that

$$\frac{1}{T_{1/k, \theta/k}} \stackrel{d}{=} k^k \prod_{j=1}^{k-1} G_{\frac{\theta+j}{k}}$$

distribution. One may obtain further representations by using the known fact[see Chaumont and Yor (2003, p. 113), that  $1/S_{1/k} \stackrel{d}{=} k^k \prod_{j=1}^{k-1} G_{j/k}$ .

PROOF. The result may be deduced from the equivalence  $G_{\theta}^k S_{1/k} = G_{\theta/k} M_{\theta/k}(\varrho_{1/k})$  established in Theorem 4.1 and the following identity

$$G_{\theta}^k = (G_{k(\frac{\theta}{k})})^k = k^k G_{\frac{\theta}{k}} \prod_{j=1}^{k-1} G_{(\theta+j)/k},$$

which is found in Chaumont and Yor (2003, p. 113).  $\square$

REMARK 18. As mentioned in Section 3, it is known from Cifarelli and Melilli (2000, p. 1394) that  $M_{\theta/2}(\varrho_{1/2}) \stackrel{d}{=} G_{(\theta+1)/2}/G_{1/2}$ , for  $\theta \geq 2$ . However the other results in Proposition 4.4 are apparently new for integer values,  $k > 2$ , and all  $\theta > 0$ .

### 4.3 Results for tilted Linnik laws and occupation times

We now address the case of the tilted processes which connects with the occupation times.

**Proposition 4.5** *Suppose that  $Z$  is  $GGC^{(b,c)}(\alpha\theta, \varrho_{\alpha})$ . Then, its Lévy measure may be read from Propositions 3.1. and 3.2 In particular a  $GGC^{(c,c)}(\alpha\theta, \varrho)$  is a  $GGC(\alpha\theta, \Lambda_{\alpha,p})$  with Lévy measure expressible as*

$$\alpha\theta s^{-1} \mathbb{E}[e^{-s/A_{\alpha,p}}]$$

Additionally for  $L_c$  a  $GGC(\alpha\theta, \Lambda_{\alpha,p})$  random variable, we have the following distributional equivalences

- (i) For all  $\theta > 0$ ,  $L_c \stackrel{d}{=} G_{\theta\alpha} P_{\alpha, \alpha\theta}(C) \stackrel{d}{=} G_{\theta\alpha} M_{\alpha\theta}(\Lambda_{\alpha,p})$ , hence  $M_{\alpha\theta}(\Lambda_{\alpha,p}) \stackrel{d}{=} P_{\alpha, \theta}(C)$
  - (ii) As a special case, when  $\theta = 1$ ,  $A_{\alpha,p}^{br} \stackrel{d}{=} M_{\alpha}(\Lambda_{\alpha,p})$
  - (iii) When  $\theta = 1$ ,  $L_c \stackrel{d}{=} G_1 A_{\alpha,p}^* \stackrel{d}{=} G_1 B_{\alpha, 1-\alpha} M_{\alpha}(\Lambda_{\alpha,p}) \stackrel{d}{=} G_1 B_{\alpha, 1-\alpha} P_{\alpha, \alpha}(C)$ , where  $A_{\alpha,p}^* \stackrel{d}{=} B_{\alpha, 1-\alpha} M_{\alpha}(\Lambda_{\alpha,p}) \stackrel{d}{=} B_{\alpha, 1-\alpha} P_{\alpha, \alpha}(C) \stackrel{d}{=} G_1 M_1(\Lambda_{\alpha,p}^{(\alpha)})$ , has density
- $$(20) \quad \frac{(1-x)}{1-p} \Lambda_{\alpha,p}(dx)$$

Note  $A_{\alpha,p}^*$  equates with the random variable of Pitman and Yor(1997b, Proposition 15).



PROOF. The correspondence to the  $P_{\alpha,\alpha\theta}(C)$  models follows from (12) as mentioned earlier. All other results are consequences of Proposition 3.2. and the use of Beta Gamma calculus  $\square$

REMARK 19. From Pitman and Yor (1997b) and Barlow, Pitman and Yor (1989, p. 307), let  $g_\alpha = \sup\{t : t \leq 1 : B_t^{(\alpha)} = 0\}$ , where it is known that  $g_\alpha \stackrel{d}{=} B_{\alpha,1-\alpha}$ . Proposition 15 in Pitman and Yor (1997b) and their subsequent discussion establish the results  $A_{\alpha,p}(g_\alpha) \stackrel{d}{=} g_\alpha A_{\alpha,p}^{br}$  with density (20). Our result shows this random variable is equivalent in distribution to the Dirichlet mean functional,  $M_1(\Lambda_{\alpha,p}^{(\alpha)}) \stackrel{d}{=} B_{\alpha,1-\alpha} M_\alpha(\Lambda_{\alpha,p})$ . Note that in view of Proposition 3.2. this can be seen to arise by first noting again that  $M_1(\varrho_\alpha^{(\alpha)}) \stackrel{d}{=} B_{\alpha,1-\alpha} M_\alpha(\varrho_\alpha) \stackrel{d}{=} X_\alpha$  where  $cX_\alpha/(cX_\alpha + 1) \stackrel{d}{=} A_{\alpha,p}$  and then applying the change of measure to get the distribution of  $M_1(\Lambda_{\alpha,p}^{(\alpha)})$ . As mentioned in section 3, the construction of Theorem 3.1, and indeed Theorem 3.2, were devised in part to understand the mechanics of the result of Pitman and Yor (1997b) in more generality. The next result extends this idea.

As mentioned in the previous remark the next result provides an extension of the distributional result of Pitman and Yor (1997b, Proposition 15). It also will yield our first concrete example of the *occupation time* of a randomly skewed Bessel process. This may also be seen as a precursor to the quantities we will encounter in Section 5.

**Proposition 4.6** For  $0 < \theta < 1$ , and  $G_\theta$  and  $G_{1-\theta}$  independent, define  $\xi_\theta^* \stackrel{d}{=} pB_{\theta,1-\theta}/([q + pB_{\theta,1-\theta}]) \stackrel{d}{=} pG_\theta/(G_\theta + G_{1-\theta}q)$ . Then define

$$(21) \quad A_{\alpha,\xi_\theta^*} \stackrel{d}{=} \frac{cB_{\theta,1-\theta}^{1/\alpha} X_\alpha}{cB_{\theta,1-\theta}^{1/\alpha} X_\alpha + 1} = \frac{cM_1(\varrho_\alpha^{(\alpha\theta)})}{cM_1(\varrho_\alpha^{(\alpha\theta)}) + 1}$$

The explicit density of the quantity in (21) is given by,

$$f_{\alpha,\theta}(y) = \frac{q^\theta}{\pi} \frac{y^{\alpha\theta-1}(1-y)^{-1} \sin(\theta C_\alpha(\frac{q^{1/\alpha}y}{p^{1/\alpha}(1-y)}))}{[y^{2\alpha}q^2 + 2qpy^\alpha(1-y)^\alpha \cos(\alpha\pi) + (1-y)^{2\alpha}p^2]^{\theta/2}}.$$

The density of  $B_{\alpha\theta,1-\alpha\theta}P_{\alpha,\alpha\theta}(C) \stackrel{d}{=} B_{\alpha\theta,1-\alpha\theta}A_{\alpha,p}^{(\alpha,\alpha\theta)} \stackrel{d}{=} B_{\alpha\theta,1-\alpha\theta}M_{\alpha\theta}(\Lambda_{\alpha,p}) \stackrel{d}{=} M_1(\Lambda_{\alpha,p}^{(\alpha\theta)})$ , is

$$(22) \quad \frac{1-y}{(1-p)^\theta} f_{\alpha,\theta}(y) = \frac{1}{\pi} \frac{y^{\alpha\theta-1} \sin(\theta C_\alpha(\frac{q^{1/\alpha}y}{p^{1/\alpha}(1-y)}))}{[y^{2\alpha}q^2 + 2qpy^\alpha(1-y)^\alpha \cos(\alpha\pi) + (1-y)^{2\alpha}p^2]^{\theta/2}}.$$

More generally the results hold for all  $0 < \theta\alpha < 1$ , where  $f_{\alpha,\theta}(y)$  is the density of the random variable  $cM_1(\varrho_\alpha^{(\alpha\theta)})/(cM_1(\varrho_\alpha^{(\alpha\theta)}) + 1)$ , and (22) is the density of  $M_1(\Lambda_{\alpha,p}^{(\alpha\theta)})$ .

PROOF. The density of (21) is obtained from Proposition 4.3. To obtain, (22) note that we are using the fact  $G_{\alpha\theta}P_{\alpha,\alpha\theta}(C) \stackrel{d}{=} G_1B_{\alpha\theta,1-\alpha\theta}P_{\alpha,\alpha\theta}(C)$ . Now it becomes evident that this distribution is obtained from exponentially tilting  $(p/q)^{1/\alpha}G_1M_1(\varrho_\alpha^{(\alpha\theta)})$  in the sense of Proposition 3.2 and Proposition 4.3.  $\square$

REMARK 20. At this point we could use Theorem 4.2, combined with Proposition 3.2, to obtain expressions for the density and cdf of  $A_{\alpha,p}^{(\alpha,\alpha\theta)} \stackrel{d}{=} P_{\alpha,\alpha\theta}(C) = M_{\alpha\theta}(\Lambda_{\alpha,p})$ . Or one could use directly (16) and (15). These would provide alternative expressions for  $P_{\alpha,\alpha\theta}(C)$  obtained in James, Lijoi and Prünster (2006). In that work, the authors addressed the case of more general functionals  $P_{\alpha,\alpha\theta}(g)$ , where they obtained these laws by a direct inversion of the appropriate Cauchy-Stieltjes transform. That work does not address functionals such as  $M_{\alpha\theta}(\varrho_\alpha)$ . A description of the laws of  $A_{\alpha,p}^{(\alpha,\alpha\theta)}$  will appear as a special case of the forthcoming Proposition 5.11.

## 5 GGC/FGGC stable compositions and occupation laws for randomly skewed Bessel processes

Raising things to the level of processes we recall the known fact that a positive Linnik process equates to  $L_{\alpha,\theta}(t) \stackrel{d}{=} S_\alpha(\Gamma_\theta(t)) \stackrel{d}{=} G_{\theta t}^{1/\alpha} S_\alpha$ , where in the second equality  $S_\alpha(t)$  is a stable subordinator independent of the gamma subordinator,  $\Gamma_\theta(t)$ . Based on our previous results we now study the class of models  $S_\alpha(Z_\theta(t)) \stackrel{d}{=} [Z_\theta(t)]^{1/\alpha} S_\alpha$  where  $Z_\theta$  is a GGC/FGGC. A special case is where  $Z_\theta(t)$  is itself a stable subordinator of index  $0 < \beta < 1$ , say  $S_\beta(t)$ , which satisfies the important identity  $S_\alpha(S_\beta(1)) \stackrel{d}{=} [S_\beta]^{1/\alpha} S_\alpha \stackrel{d}{=} S_{\alpha\beta}$ , an  $(\alpha\beta)$ -stable random variable. Note that in this case  $Z_\theta = S_\beta$  is not a FGGC.

REMARK 21. Obviously compositions of a stable subordinator have been previously studied from several important perspectives. For instance, this operation has recently been shown to play an interesting role in applications involving coagulation/fragmentation phenomena as described in Pitman (2006) and Bertoin (2006).

Here, using the property that  $Z_\theta(t) \stackrel{d}{=} G_{\theta t} M_{\theta t}(\nu)$ , we take a different view of  $S_\alpha(Z_\theta(t))$  as being equivalent in distribution to  $G_{\theta t}^{1/\alpha} S_\alpha(t) [M_{\theta t}(\nu)]^{1/\alpha}$ . That is to say, the viewpoint that these are scale mixtures of positive Linnik random variables, which leads to a variety of interesting consequences. Lets call this class  $\text{GGC}_\alpha(\theta, \nu)$  which will also denote the law of the random variable  $S_\alpha(Z_\theta(1))$ .

Now, from the description of the Lévy density of a Linnik random variable in Section 4, it is evident that the Laplace transform, at  $t = 1$ , of  $S_\alpha(Z_\theta(1))$ , can be expressed as

$$\mathbb{E}[(1 + \lambda^\alpha M_\theta(\nu))^{-\theta}] = e^{-\psi_\theta^{(\alpha)}(\lambda)}$$

where

$$\psi_\theta^{(\alpha)}(\lambda) = \int_0^\infty \theta \log(1 + \lambda^\alpha r) \nu(dr) = \alpha\theta \int_0^\infty \mathbb{E}[\log(1 + \lambda X_\alpha r^{1/\alpha})] \nu(dr).$$

That is, the Lévy measure of  $S_\alpha(Z_\theta(1)) \stackrel{d}{=} G_\theta^{1/\alpha} S_\alpha [M_\theta(\nu)]^{1/\alpha}$  can be expressed as

$$\alpha\theta s^{-1} \int_0^\infty \mathbb{E}[e^{-s/(X_\alpha r^{1/\alpha})}] \nu(dr) \text{ for } s > 0.$$

### 5.1 Tilting and Randomly Skewed Bessel Bridges

In parallel to Section 4, we also discuss its tilted version, which as we shall see connects naturally with the idea of randomly skewed Bessel processes and bridges. Using the same type of exponential tilting as in Section 4 we call the resulting random variables  $\text{GGC}_\alpha^{(b,c)}(\theta, \nu)$ . Again, provided that  $c \neq 0$ , we see that if a random variable  $L$  is  $\text{GGC}_\alpha^{(b,c)}(\theta, \nu)$ , then  $L \stackrel{d}{=} (b/c)L^*$  where  $L^*$  is  $\text{GGC}_\alpha^{(c,c)}(\theta, \nu)$ . An initial description of the Laplace transform of  $L^*$  is

$$(23) \quad \frac{\mathbb{E}[(1 + (1 + \lambda)^\alpha c^\alpha M_\theta(\nu))^{-\theta}]}{\mathbb{E}[(1 + c^\alpha M_\theta(\nu))^{-\theta}]} = e^{-\Psi_\theta^{(\alpha)}(\lambda)}$$

where  $\Psi_\theta^{(\alpha)}(\lambda) = \psi_\theta^{(\alpha)}(c(1 + \lambda)) - \psi_\theta^{(\alpha)}(c)$ . Now, by setting  $p_\alpha(r) = c^\alpha r / (1 + c^\alpha r)$  and algebra, this is equivalent to

$$(24) \quad \Psi_\theta^{(\alpha)}(\lambda) = \alpha\theta \int_0^\infty \mathbb{E}[\log(1 + \lambda A_{\alpha,p_\alpha(r)})] \nu(dr) = \alpha\theta \int_0^\infty \log(1 + \lambda x) \Lambda_{\alpha,\nu}(dx)$$

where

$$\Lambda_{\alpha,\nu}(dx) = \int_0^\infty \Lambda_{\alpha,p_\alpha(r)}(dx)\nu(dr).$$

Before examining this quantity further we first formally define, albeit briefly, what we mean by randomly skewed Bessel processes and bridges and their occupation times. Let  $\xi$  denote a random variable on  $[0, 1]$  chosen independently of a Bessel process  $\{B^{(\alpha)}(t) : t \geq 0\}$ . Then the process  $B_\xi^{(\alpha)}(t)$  is said to be a  $\xi$ -randomly skewed Bessel process if  $B_\xi^{(\alpha)}|_{\xi=p}$  is a  $p$ -skewed Bessel process as defined in Barlow, Pitman and Yor (1989). That is,  $\mathbb{P}(B_\xi^{(\alpha)}(s) > 0|\xi) = \xi$ . Say that  $A_{\alpha,\xi}(t) := \int_0^t \mathbb{I}(B_\xi^{(\alpha)}(s) > 0)ds$  is the time spent positive of a  $\xi$ -randomly skewed Bessel process up to time  $t$ . Then conditional on  $\xi = p$ , it has distribution  $A_{\alpha,p}(t)$ . Equivalently, one has, for  $A_{\alpha,\xi} := A_{\alpha,\xi}(1)$ ,

$$(25) \quad A_{\alpha,\xi} \stackrel{d}{=} \frac{\xi^{1/\alpha} X_\alpha}{\xi^{1/\alpha} X_\alpha + (1-\xi)^{1/\alpha}}.$$

Based on (25) we present an interesting special case.

**Proposition 5.1** Define  $p_\alpha := p^\alpha/[p^\alpha + q^\alpha]$ . For  $0 < \beta < 1$ , let  $\xi \stackrel{d}{=} A_{\beta,p_\alpha} = c^\alpha X_\beta/(c^\alpha X_\beta + 1)$  which is equivalent to  $\xi/(1-\xi) = c^\alpha X_\beta$ . Then

$$A_{\alpha,\xi} \stackrel{d}{=} A_{\alpha\beta,p}.$$

PROOF. The result follows from the definition in (25) and the easily verified fact that  $[X_\beta]^{1/\alpha} X_\alpha \stackrel{d}{=} X_{\alpha\beta}$   $\square$

$\xi$ -randomly skewed bridges are defined in an analogous manner where conditionally on  $\xi = p$  they are  $p$ -skewed Bessel bridges. Denote the random variable corresponding to the time spent positive up till time 1 of such a process as  $A_{\alpha,\xi}^{br}$ , which conditional on  $\xi = p$ , is equivalent in distribution to  $A_{\alpha,p}^{br} \stackrel{d}{=} P_{\alpha,\alpha}(C)$ , having the Cauchy-Stieltjes transform of order  $\alpha\theta$  in (12). More generally, by the usual  $\text{PD}(\alpha, \alpha\theta)$  change of measure, we can define  $A_{\alpha,\xi}^{(\alpha,\alpha\theta)}$  which conditionally on  $\xi = p$  equates in distribution with  $A_{\alpha,p}^{(\alpha,\alpha\theta)} \stackrel{d}{=} P_{\alpha,\alpha\theta}(C)$ . Now we shall describe a sub-class of such processes which equates with a random variable described by the transform (23).

**Proposition 5.2** Let  $A_{\alpha,\xi_\theta}^{(\alpha,\alpha\theta)}$  correspond to a  $\xi_\theta$ -skewed occupation time with  $\xi_\theta$  having the specific density  $Q_\theta(du|\nu)/du = \kappa_\theta(1-u)^\theta \tilde{Q}_\theta(du|\nu)/du$  where  $\kappa_\theta = 1/\mathbb{E}[(1+c^\alpha M_\theta(\nu))^{-\theta}] = e^{\psi_\theta^{(\alpha)}(c)}$  and

$$\tilde{Q}_\theta(du) = \frac{1}{(1-u)^2} f_{c^\alpha M_\theta}(\frac{u}{1-u}|\nu) du$$

That is  $\tilde{Q}_\theta$  is the distribution of the random variable  $c^\alpha M_\theta(\nu)/(c^\alpha M_\theta(\nu) + 1)$ . Equivalently  $\xi_\theta \stackrel{d}{=} M_\theta(\nu^{(c^\alpha, c^\alpha)})$  and  $A_{\alpha,\xi_\theta}^{(\alpha,\alpha\theta)}$  satisfies the following properties.

(i) The Cauchy-Stieltjes transform of order  $\alpha\theta$  of  $A_{\alpha,\xi_\theta}^{(\alpha,\alpha\theta)}$  is expressible as

$$\int_0^1 \mathbb{E}[(1 + \lambda A_{\alpha,p}^{(\alpha,\alpha\theta)})^{-\alpha\theta}] Q_\theta(dp|\nu) = \int_0^1 (q + p(1+\lambda)^\alpha)^{-\theta} Q_\theta(dp|\nu)$$

and equals (23).

(ii) Hence, by (24),  $A_{\alpha,\xi_\theta}^{(\alpha,\alpha\theta)} \stackrel{d}{=} M_{\alpha\theta}(\Lambda_{\alpha,\nu})$ , and  $L \stackrel{d}{=} G_{\alpha\theta} A_{\alpha,\xi_\theta}^{(\alpha,\alpha\theta)}$  is  $\text{GGC}(\alpha\theta, \Lambda_{\alpha,\nu})$ .

PROOF. This result is easily verified by an argument similar to the proof of Theorem 3.2. Hence we omit the details.  $\square$

Now define

$$(26) \quad \varrho_{\alpha, \nu}(x) = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \frac{x^{\alpha-1} r}{x^{2\alpha} + 2x^\alpha r \cos(\alpha\pi) + r^2} \nu(dr).$$

We can summarize these results in an equivalent manner;

**Proposition 5.3** *Let  $L \stackrel{d}{=} [Z_\theta(1)]^{1/\alpha} S_\alpha$  denote a  $GGC_\alpha(\theta, \nu)$  random variable then*

$$(27) \quad L \stackrel{d}{=} G_{\theta\alpha} M_{\alpha\theta}(\varrho) [M_\theta(\nu)]^{1/\alpha} = G_{\alpha\theta} M_{\alpha\theta}(\varrho_{\alpha, \nu}).$$

where  $\varrho_{\alpha, \nu}$  is defined in (26). That is  $GGC_\alpha(\theta, \nu)$  is equivalent to  $GGC(\alpha\theta, \varrho_{\alpha, \nu})$ . Now without loss of generality if  $c \neq 0$ , then set  $c = 1$  and consider the case where  $\tilde{L}$  is  $GGC_\alpha(\theta, \nu^{(1,1)})$  with Laplace transform as in (23). Then  $\tilde{L} \stackrel{d}{=} G_{\alpha\theta} A_{\alpha, \xi_\theta}^{(\alpha, \alpha\theta)} \stackrel{d}{=} G_{\alpha\theta} M_{\alpha\theta}(\Lambda_{\alpha, \nu})$  is  $GGC(\alpha\theta, \Lambda_{\alpha, \nu})$ . In general if  $L$  is  $GGC_\alpha^{(b,c)}(\theta, \nu)$  random variable, then its Lévy measure can be expressed as,

$$\alpha\theta s^{-1} e^{-(c/b)s} \int_0^\infty \mathbb{E}[e^{-s/[bX_\alpha r^{1/\alpha}]}] \nu(dr). \square$$

The next result is specialized to the case where  $\nu = H$ .

**Proposition 5.4** *Suppose that  $L$  is  $GGC(\alpha\theta, \varrho_{\alpha, H})$ , then there exists a random variable  $R$  with distribution  $H$ , such that  $R^{1/\alpha} X_\alpha$  has density  $\varrho_{\alpha, H}$  and  $L$  is a FGGC with Lévy measure*

$$\alpha\theta s^{-1} \mathbb{E}[e^{-s/[R^{1/\alpha} X_\alpha]}].$$

Correspondingly  $\tilde{L}$ , the exponential tilt of  $L$  with  $c = 1$ , is  $GGC(\alpha\theta, \Lambda_{\alpha, H})$  and has the Lévy measure

$$\alpha\theta s^{-1} \mathbb{E}[e^{-s/A_{\alpha, \xi}}]$$

where  $\xi/(1-\xi) \stackrel{d}{=} R$  and  $A_{\alpha, \xi} \stackrel{d}{=} R^{1/\alpha} X_\alpha / (R^{1/\alpha} X_\alpha + 1)$  has distribution  $\Lambda_{\alpha, H}$ . Hence in this setting  $A_{\alpha, \xi_\theta}^{(\alpha, \alpha\theta)} \stackrel{d}{=} M_{\alpha\theta}(\Lambda_{\alpha, H})$ , and  $\xi_\theta = M_\theta(H^{(c^\alpha, c^\alpha)})$  are Dirichlet mean functionals.  $\square$

REMARK 22. Bondesson (1992) describes various features of the class which we call  $GGC_\alpha(\theta, \nu)$ . See for instance Theorem 3.3.2 of that work which establishes the fact that this class of models are indeed GGC. As noted by Bondesson (1992), compositions of GGC random variables with some GGC subordinator are not always GGC. An example is the composition of a Gamma subordinator with another GGC process. Note that our representation of the Lévy measure in terms of  $X_\alpha R^{1/\alpha}$  and our subsequent usage of it appears to be new.

REMARK 23. It is important to note that special cases of such models have already appeared in the literature. Proposition 19 of Pitman (1999), in connection with coagulation phenomena, shows, with obvious rephrasing, that the law of  $A_{\alpha, \xi}^{br}$ , where  $\xi = B_{\beta, 1-\beta}$ , corresponds to a  $BETA(\alpha - \alpha\beta, \alpha\beta)$  random variable. Another interesting case where a model having a particular distribution of  $A_{\alpha, \xi}^{br}$  may be found is in Aldous and Pitman (2004). Specifically, the distribution of a random lengths  $1 - T_k$  of a  $T$ -partition, as described in equation (67) of Aldous and Pitman (2004), may be re-expressed as

$$\mathbb{P}(T_k \in dt) = \int_0^1 \mathbb{P}(A_{\alpha, p}^{br} \in dx) \Pi_k(dp) = \mathbb{P}(A_{\alpha, \xi}^{br} \in dx)$$

where  $\Pi_k(dp)/dp = (-\log(p))^{k-1}/(k-1)!$  and hence  $\xi \stackrel{d}{=} \prod_{i=1}^k U_i$ , for  $(U_i)$  independent Uniform $[0, 1]$  random variables. We now show how to use a construction of Aldous and Pitman (2004) that equates, more generally, with the distribution of  $A_{\alpha, \xi}^{(\alpha, \alpha\theta)}$ . Extending the definition in Aldous and Pitman (2004, p.24), define for  $0 < u < 1$

$$\tau_{\alpha, u}^{(\alpha, \alpha\theta)} = \inf\{t : \frac{\ell_t^{(\alpha, \alpha\theta)}}{\ell_1^{(\alpha, \alpha\theta)}} = u\}.$$

Then it follows that  $\tau_{\alpha, u}^{(\alpha, \alpha\theta)} \stackrel{d}{=} A_{\alpha, u}^{(\alpha, \alpha\theta)}$  and hence,  $\tau_{\alpha, \xi}^{(\alpha, \alpha\theta)} \stackrel{d}{=} A_{\alpha, \xi}^{(\alpha, \alpha\theta)}$ .

REMARK 24. We note that one could use the results for the laws of  $A_{\alpha, p}^{(\alpha, \alpha\theta)} \stackrel{d}{=} P_{\alpha, \alpha\theta}(C)$  in James, Lijoi and Prünster (2006), or the form obtainable from Section 4, to represent the laws of  $A_{\alpha, \xi}^{(\alpha, \alpha\theta)}$  by mixing with respect to a distribution of  $\xi$ . However it is not a trivial matter to simplify such expressions. Here we will use a direct approach.

## 5.2 Random time changes for Local and Occupation times

Here we translate the results in the previous section in terms of random time changes of the local and occupation times.

**Proposition 5.5** *Let  $(\ell_t^{(\alpha, \alpha\theta)}; t \geq 0)$  denote the local time under the  $PD(\alpha, \alpha\theta)$  law for  $\alpha\theta > 0$ . Let  $Z_\theta(1)$  denote a  $GGC(\theta, \nu)$  random variable. Then it follows that*

$$\begin{aligned} (i) \quad & \ell_{G_{\theta\alpha} M_\theta^{1/\alpha}(\nu)}^{(\alpha, \alpha\theta)} \stackrel{d}{=} \ell_{G_{\theta\alpha}}^{(\alpha, \alpha\theta)} M_\theta(\nu) \stackrel{d}{=} Z_\theta(1) \stackrel{d}{=} G_\theta M_\theta(\nu) \\ (ii) \quad & A_{\alpha, p}(\tau_\alpha(Z_\theta(1))) \stackrel{d}{=} p^{1/\alpha} [\ell_{G_{\theta\alpha}}^{(\alpha, \alpha\theta)}]^{1/\alpha} S_\alpha M_\theta^{1/\alpha}(\nu) \stackrel{d}{=} p^{1/\alpha} \tilde{L}_{\alpha, \theta}. \end{aligned}$$

PROOF. Statement [(i)] follows from the scaling property in (17), (27), and the fact that  $M_{\alpha\theta}(\varrho_\alpha) \stackrel{d}{=} [\ell_{S_\alpha}^{(\alpha, \alpha\theta)}]^{1/\alpha}$ . Statement [(ii)] is straightforward.  $\square$

The result states that we can arrange for the time changed local times to have any GGC distribution. We present the next result for a simple illustration

**Proposition 5.6** *Let  $(\ell_t^{(\alpha, \alpha\theta)}; t \geq 0)$  denote the local time under the  $PD(\alpha, \alpha\theta)$  law for  $\alpha\theta > 0$ . Set  $M_\theta(\nu) = B_{\alpha\theta, \theta(1-\alpha)}$ . Then it follows that*

$$\begin{aligned} (i) \quad & \ell_{G_{\theta\alpha} B_{\alpha\theta, \theta(1-\alpha)}^{1/\alpha}}^{(\alpha, \alpha\theta)} = \ell_{G_{\theta\alpha}}^{(\alpha, \alpha\theta)} B_{\alpha\theta, \theta(1-\alpha)} \stackrel{d}{=} G_{\alpha\theta} \\ (ii) \quad & A_{\alpha, p}(\tau_\alpha(G_{\alpha\theta})) \stackrel{d}{=} p^{1/\alpha} G_{\alpha\theta}^{1/\alpha} S_\alpha \end{aligned}$$

## 5.3 Scaling Calculus for $GGC_\alpha(\theta, \nu)$ mean functionals

We now show that our results establish a rather remarkable type of scaling calculus for mean functionals.

**Proposition 5.7** *If  $L$  is  $GGC_\alpha^{(b, c)}(\theta, \nu)$ , it follows that  $GGC_\alpha^{(1, 0)}(\theta, \nu) = GGC(\alpha\theta, \varrho_{\alpha, \nu})$  and  $GGC_\alpha^{(1, 1)}(\theta, \nu) = GGC(\alpha\theta, \Lambda_{\alpha, \nu})$ . That is in the first case  $L \stackrel{d}{=} G_{\alpha\theta} M_{\alpha\theta}(\varrho_{\alpha, \nu})$  and in the second  $L \stackrel{d}{=} G_{\alpha\theta} M_{\alpha\theta}(\Lambda_{\alpha, \nu})$ . These points lead to the following results*

$$(i) \quad M_{\alpha\theta}(\varrho_{\alpha, \nu}) \stackrel{d}{=} M_{\alpha\theta}(\varrho_\alpha) [M_\theta(\nu)]^{1/\alpha}$$

(ii) Following Theorem 3.2 and Proposition 3.1,  $A_{\alpha, \xi_\theta}^{(\alpha, \alpha\theta)} = M_{\alpha\theta}(\Lambda_{\alpha, \nu}) \stackrel{d}{=} Y_{\alpha\theta, 1}$ , with  $M \stackrel{d}{=} M_{\alpha\theta}(\varrho_{\alpha, \nu})$ .

The results apply for arbitrary Thorin measures  $\nu$  but yield results specific to Dirichlet mean functionals by setting  $\nu = H$ .  $\square$

A variation of this result is given below,

**Proposition 5.8** *Theorem 3.1 and Proposition 5.7 imply,*

$$(i) \quad M_1(\varrho_{\alpha, H}^{(\alpha\theta)}) = B_{\alpha\theta, 1-\alpha\theta} M_{\alpha\theta}(\varrho_{\alpha, H}) \stackrel{d}{=} M_1(\varrho_{\alpha}^{(\alpha\theta)}) [M_\theta(H)]^{1/\alpha}$$

$$(ii) \quad M_1(\varrho_{\alpha, H}^{(\alpha)}) = X_\alpha [M_1(H)]^{1/\alpha}$$

$$(ii) \quad M_1(\Lambda_{\alpha, H}^{(\alpha\theta)}) \stackrel{d}{=} B_{\alpha\theta, 1-\alpha\theta} A_{\alpha, \xi}^{(\alpha, \alpha\theta)}.$$

Notice that when  $0 < \theta < 1$ ,  $M_1(\varrho_{\alpha, H}^{(\alpha\theta)}) \stackrel{d}{=} X_\alpha B_{\theta, 1-\theta}^{1/\alpha} [M_\theta(H)]^{1/\alpha}$  but  $B_{\theta, 1-\theta}^{1/\alpha} [M_\theta(H)]^{1/\alpha} = [M_1(H^{(\theta)})]^{1/\alpha}$ . Hence this is a special case of statement [(ii)].  $\square$

Notice now that

$$\varrho_{\alpha, H}(x) = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \frac{x^{\alpha-1} r}{x^{2\alpha} + 2x^\alpha r \cos(\alpha\pi) + r^2} H(dr).$$

The cdf of  $X_\alpha R^{1/\alpha}$ ,  $\Upsilon_\alpha(x) = \mathbb{P}(X_\alpha R^{1/\alpha} \leq x)$ , can be represented as

$$(28) \quad \Upsilon_\alpha(x) = 1 - \frac{1}{\pi\alpha} \int_0^\infty \cot^{-1} \left( \cot(\pi\alpha) + \frac{x^\alpha/r}{\sin(\pi\alpha)} \right) H(dr)$$

which simplifies in the case of  $\alpha = 1/2$  to,

$$\Upsilon_{1/2}(x) = \frac{2}{\pi} \int_0^\infty \arctan \left( \frac{\sqrt{x}}{r} \right) H(dr).$$

**Proposition 5.9** *Let  $X_\alpha = S_\alpha/S'_\alpha$ , having density (8). Let  $R$  be a random variable with distribution  $H$ . Then the random variable  $X_\alpha R^{1/\alpha}$  has density  $\varrho_{\alpha, H}$ . Define  $\mathcal{S}_\alpha^H(x) = \mathbb{E}[\log(|x - X_\alpha R^{1/\alpha}|) \mathbb{I}(x \neq X_\alpha R^{1/\alpha})]$ . Then,*

$$\mathcal{S}_\alpha^H(x) = \frac{1}{2\alpha} \int_0^\infty \log(x^{2\alpha} + 2x^\alpha r \cos(\alpha\pi) + r^2) H(dr)$$

with derivative

$$\sigma_\alpha^H(x) = \int_0^\infty \frac{x^{2\alpha-1} + x^{\alpha-1} r \cos(\alpha\pi)}{x^{2\alpha} + 2x^\alpha r \cos(\alpha\pi) + r^2} H(dr).$$

$$\mathcal{S}_{1/2}^H(x) = \int_0^\infty \log(x + r^2) H(dr) = \int_0^\infty \log(1 + r^2/x) H(dr) + \log(x). \quad \square$$

The next result shows that  $\mathcal{S}_\alpha^H(x)$  may be seen as the Lévy exponent of certain FGDC models. This provides another way for further simplification.

**Proposition 5.10** *Let  $R$  be a random variable with distribution  $H$ . Define  $W_{\alpha, x} \stackrel{d}{=} 2Rx^{-\alpha} \cos(\pi\alpha) + R^2 x^{-2\alpha}$  and denote its distribution as  $H_\alpha^x$ . Then,*

$$e^{-\theta\alpha\mathcal{S}_\alpha^H(x)} = \mathbb{E}[e^{-G_{\theta/2} M_{\theta/2}(H_\alpha^x)}] x^{-\theta\alpha}$$

When  $\alpha = 1/2$  let  $H_{1/2}(y) = H(\sqrt{y})$  denote the cdf of  $R^2$  then

$$e^{-\theta/2\mathcal{S}_{1/2}^H(x)} = \mathbb{E}[e^{-\frac{1}{x} G_{\theta/2} M_{\theta/2}(H_{1/2})}] x^{-\theta/2}.$$

$\square$

## 5.4 Density formula

With this we can obtain explicit expressions for the cdf and density of  $M_{\alpha\theta}(\varrho_{\alpha,H})$  as follows

**Theorem 5.1** Recall that  $M_{\alpha\theta}(\varrho_{\alpha,H}) \stackrel{d}{=} M_{\alpha\theta}(\varrho_{\alpha})[M_{\theta}(H)]^{1/\alpha}$ . The form of the cdf for  $M_{\alpha\theta}(\varrho_{\alpha,H})$  for all  $\alpha\theta > 0$ , is given by (1), with  $\theta := \alpha\theta$ , and

$$\Delta_{\theta\alpha}(x|\varrho_{\alpha,H}) = \frac{1}{\pi} \sin(\pi\theta\alpha\Upsilon_{\alpha}(x))e^{-\theta\alpha\mathcal{J}_{\alpha}^H(x)}$$

where  $\Upsilon_{\alpha}$  is given in (28).  $\Delta_1(x|\varrho_{\alpha,H})$  is the density of  $M_1(\varrho_{\alpha,H})$ . Furthermore, a general expression for the density is obtained from (3) with  $\theta := \alpha\theta$  and

$$d_{\alpha\theta}(x|\varrho_{\alpha,H}) = \frac{\alpha\theta}{\pi} e^{-\theta\alpha\mathcal{J}_{\alpha}^H(x)} [\sin(\pi\alpha[1 - \theta\Upsilon_{\alpha}(x)])B_{\alpha,1}^H(x) - \sin(\pi\theta\alpha\Upsilon_{\alpha}(x))B_{\alpha,2}^H(x)],$$

where

$$B_{\alpha,1}^H(x) = \int_0^{\infty} \frac{x^{\alpha-1}r}{x^{2\alpha} + 2x^{\alpha}r \cos(\alpha\pi) + r^2} H(dr)$$

and

$$B_{\alpha,2}^H(x) = \int_0^{\infty} \frac{x^{2\alpha-1}}{x^{2\alpha} + 2x^{\alpha}r \cos(\alpha\pi) + r^2} H(dr).$$

□

Now applying Theorem 3.1 we obtain,

**Theorem 5.2** For  $0 < \alpha\theta < 1$ , the density of  $M_1(\varrho_{\alpha,H}^{(\alpha\theta)}) = B_{\alpha\theta,1-\alpha\theta}M_{\alpha\theta}(\varrho_{\alpha})[M_{\theta}(H)]^{1/\alpha}$  is given by

$$\frac{1}{\pi} \sin(\pi\theta\alpha[1 - \Upsilon_{\alpha}(x)])e^{-\theta\alpha\mathcal{J}_{\alpha}^H(x)}x^{\alpha\theta-1}$$

When  $\theta = 1$ , the density of  $M_1(\varrho_{\alpha,H}^{(\alpha)}) \stackrel{d}{=} X_{\alpha}[M_1(H)]^{1/\alpha}$  can also be expressed as,

$$\varrho_{\alpha,F_{M_1(H)}}(x) = \frac{\sin(\pi\alpha)}{\pi} \int_0^{\infty} \frac{x^{\alpha-1}r}{x^{2\alpha} + 2x^{\alpha}r \cos(\alpha\pi) + r^2} f_{M_1}(r|H)dr.$$

□

Then next results are for the  $\xi$ -skewed occupation times.

**Proposition 5.11** Assume without loss of generality that  $c = 1$ . Then, for all  $\alpha\theta > 0$ , the density of  $M_{\alpha\theta}(\Lambda_{\alpha,H}) \stackrel{d}{=} A_{\alpha,\xi_{\theta}}^{(\alpha,\alpha\theta)}$  can be expressed as

$$\kappa_{\theta}(1-y)^{\alpha\theta-2}f_{M_{\alpha\theta}}\left(\frac{y}{1-y}|\varrho_{\alpha,H}\right)$$

PROOF. This is an application of Proposition 3.2. See also statement [(ii)] of Proposition 5.7. □

The next result is a generalization of Proposition 4.6.

**Proposition 5.12** Assume without loss of generality that  $c = 1$ . Then, for all  $0 < \alpha\theta < 1$ , the density of  $M_1(\Lambda_{\alpha,H}^{(\alpha\theta)}) \stackrel{d}{=} B_{\alpha\theta,1-\alpha\theta}A_{\alpha,\xi_{\theta}}^{(\alpha,\alpha\theta)}$  can be expressed as

$$\kappa_{\theta}\frac{1}{\pi} \sin(\pi\theta\alpha[1 - \Upsilon_{\alpha}(\frac{y}{1-y})])e^{-\theta\alpha\mathcal{J}_{\alpha}^H(\frac{y}{1-y})}y^{\alpha\theta-1}(1-y)^{-\alpha\theta}$$

PROOF. Apply Proposition 3.2 to the density of  $M_1(\varrho_{\alpha,H}^{(\alpha\theta)})$  given in Theorem 5.2  $\square$

The next result is also immediate from Theorem 5.2

**Proposition 5.13** *Set  $\xi/(1-\xi) \stackrel{d}{=} M_1(H)$ , then the density of*

$$A_{\alpha,\xi} \stackrel{d}{=} \frac{[M_1(H)]^{1/\alpha} X_\alpha}{[M_1(H)]^{1/\alpha} X_\alpha + 1}$$

is

$$\frac{1}{\pi} \sin(\pi\alpha[1 - \Upsilon_\alpha(\frac{y}{1-y})]) e^{-\alpha \mathcal{J}_\alpha^H(\frac{y}{1-y})} y^{\alpha-1} (1-y)^{-\alpha-1}$$

$\square$

REMARK 25. We mention briefly the following example, in the case of  $\alpha = 1/2$ . Choose  $R^2 = G_1/E$ , then  $R^2 X_{1/2}$  has the distribution of a Pareto TYPE III law with cdf

$$\Upsilon_{1/2}(x) = 1 - (1 + x^{1/2})^{-1} = \frac{\sqrt{x}}{\sqrt{x} + 1}$$

and  $e^{-\theta/2 \mathcal{J}_{1/2}^H(x)} = \mathbb{E}[e^{-\frac{1}{x} G_{\theta/2} M_{\theta/2}(\zeta)}] x^{-\theta/2} = x^{\frac{x}{1-x} \frac{\theta}{2}}$ . Where the last expression is obtained from Proposition 5.10 and Proposition 3.3.

## 5.5 Identities for PD local times and skew Bessel Bridges

We close by showing how the scaling properties discussed in Proposition 5.7 and 5.8, coupled with the identity  $X_{\alpha\beta} \stackrel{d}{=} X_\alpha [X_\beta]^{1/\alpha}$ , translate into interesting identities for skew Bessel bridges and corresponding local times. The first result is an immediate consequence of Propositions 4.2 and 5.7.

**Proposition 5.14** *For  $0 < \beta < 1$ , set  $R \stackrel{d}{=} X_\beta$ , that is  $H = \varrho_\beta$ , then for  $\theta > 0$ ,*

$$M_{\alpha\theta}(\varrho_{\alpha\beta}) \stackrel{d}{=} M_{\alpha\theta}(\varrho_\alpha) [M_\theta(\varrho_\beta)]^{1/\alpha}.$$

Equivalently,

$$(i) \ell_{S_{\alpha\beta}}^{(\alpha\beta, \alpha\theta)} \stackrel{d}{=} \ell_{S_\beta}^{(\beta, \theta)} [\ell_{S_\alpha}^{(\alpha, \alpha\theta)}]^\beta$$

(ii) *In terms of the  $(\alpha)$ -diversity,*

$$\frac{S_{\alpha\beta}}{T_{\alpha\beta, \alpha\theta}} \stackrel{d}{=} \frac{S_\alpha}{T_{\alpha, \alpha\theta}} \left[ \frac{S_\beta}{T_{\beta, \theta}} \right]^{1/\alpha}$$

which implies that  $T_{\alpha\beta, \alpha\theta} \stackrel{d}{=} T_{\alpha, \alpha\theta} [T_{\beta, \theta}]^{1/\alpha}.$   $\square$

Now applying Proposition 5.8 leads to,

**Proposition 5.15** *For  $0 < \beta < 1$ , set  $R \stackrel{d}{=} X_\beta$ , that is  $H = \varrho_\beta$ , which corresponds to  $\varrho_{\alpha,H} = \varrho_{\alpha\beta}$ , then for  $0 < \alpha\theta < 1$ ,*

$$(29) \quad M_1(\varrho_{\alpha\beta}^{(\alpha\theta)}) \stackrel{d}{=} M_1(\varrho_\alpha^{(\alpha\theta)}) [M_\theta(\varrho_\beta)]^{1/\alpha}$$

Recall that  $[\ell_{S_\alpha}^{(\alpha, 0)}]^{1/\alpha} \stackrel{d}{=} X_\alpha \stackrel{d}{=} M_1(\varrho_\alpha^{(\alpha)}) \stackrel{d}{=} B_{\alpha, 1-\alpha} M_\alpha(\varrho_\alpha)$ , then as special cases of (29), with the choice of  $\theta = \beta$  and  $\theta = 1$  respectively,



- (i)  $X_{\alpha\beta} \stackrel{d}{=} M_1(\varrho_{\alpha\beta}^{(\alpha\beta)}) \stackrel{d}{=} M_1(\varrho_{\alpha}^{(\alpha\beta)})[M_{\beta}(\varrho_{\beta})]^{1/\alpha}$
- (ii)  $M_1(\varrho_{\alpha\beta}^{(\alpha)}) \stackrel{d}{=} X_{\alpha}[M_1(\varrho_{\beta})]^{1/\alpha}$ .  $\square$

The next result shows how to recover  $p$ -skew Bessel bridges from  $\xi$ -skewed bridges.

**Proposition 5.16** *For  $0 < \beta < 1$ , set  $R \stackrel{d}{=} c^{\alpha} X_{\beta}$ , where  $c = (p/q)^{1/\alpha}$ , following Proposition 5.2 this means that  $\Lambda_{\alpha,H} = \Lambda_{\alpha\beta,p}$ . Now define  $p_{\alpha} = p^{\alpha}/(p^{\alpha} + q^{\alpha})$ . Then,  $\xi_{\theta} \stackrel{d}{=} A_{\beta,p_{\alpha}}^{(\beta,\theta)} \stackrel{d}{=} M_{\theta}(\Lambda_{\beta,p_{\alpha}})$  and for this choice,*

$$(30) \quad A_{\alpha,\xi_{\theta}}^{(\alpha,\alpha\theta)} \stackrel{d}{=} A_{\alpha\beta,p}^{(\alpha\beta,\alpha\theta)} \stackrel{d}{=} M_{\alpha\theta}(\Lambda_{\alpha\beta,p}) \stackrel{d}{=} P_{\alpha\beta,\alpha\theta}(C)$$

Results recovering  $p$ -skew Bessel bridges follows as special cases of (30), with the choice of  $\theta = \beta$  and  $\theta = 1$  respectively,

- (i)  $A_{\alpha,\xi_{\beta}}^{(\alpha,\alpha\beta)} \stackrel{d}{=} A_{\alpha\beta,p}^{(\alpha\beta,\alpha\beta)}$
- (ii)  $A_{\alpha,\xi_1}^{(\alpha,\alpha)} \stackrel{d}{=} A_{\alpha\beta,p}^{(\alpha\beta,\alpha)}$

PROOF. We present a proof for clarity. First, as noted, the choice of  $R = c^{\alpha} X_{\beta}$  coupled with proposition 5.2 shows that  $\Lambda_{\alpha,H} = \Lambda_{\alpha\beta,p}$ , leading to (30). What remains is to verify that  $\xi_{\theta} \stackrel{d}{=} A_{\beta,p_{\alpha}}^{(\beta,\theta)}$ . Noting Proposition 5.4,  $\xi_{\theta} \stackrel{d}{=} M_{\theta}(H^{(c^{\alpha},c^{\alpha})})$ , but here  $H^{(c^{\alpha},c^{\alpha})} = \Lambda_{\beta,p_{\alpha}}$ , when  $H = \varrho_{\beta}$ .  $\square$

So in closing we see, as a special case, that occupation time of a Brownian bridge up to time 1, say  $A_{1/2}^{br} := A_{1/2,1/2}^{(1/2,1/2)} \stackrel{d}{=} B_{1/2,1/2}$  is equivalent in distribution to the time spent positive up to time 1 of a  $A_{\beta,1/2}^{(\beta,\beta)}$  randomly skewed process  $B_{\alpha}^{(\alpha,1/2)}(t)$  for any  $\alpha = 1/(2\beta)$  and  $1/2 < \beta < 1$ . That is to say, this process is randomly skewed by the random variable corresponding to the time spent positive up till time 1 of a Bessel bridge of dimension  $2 - 2\beta < 1$ ,  $B_{\beta}^{(\beta,\beta)}(t)$ .

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